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Yoshikazu Giga

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A Level Set Approach

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To my parents, Kazuyo and Kenjiro

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Preface

This book is intended to be a self-contained introduction to analytic foundations of a level set method for various surface evolution equations including curvature flow equations. These equations are important in various fields including material sciences, image processing and differential geometry. The goal of this book is to introduce a generalized notion of solutions allowing singularities and solve the initial-value problem globally-in-time in a generalized sense. Various equivalent definitions of solutions are studied. Several new results on equivalence are also presented.

We present here a rather complete introduction to the theory of viscosity solutions which is a key tool for the level set method. Also a self-contained explanation is given for general surface evolution equations of the second order. Although most of the results in this book are more or less known, they are scattered in several references, sometimes without proof. This book presents these results in a synthetic way with full proofs. However, the references are not exhaustive at all.

The book is suitable for applied researchers who would like to know the detail of the theory as well as its flavour. No familiarity with differential geometry and the theory of viscosity solutions is required. The prerequisites are calculus, linear algebra and some familiarity with semicontinuous functions. This book is also suitable for upper level under graduate students who are interested in the field.

I am grateful to Professor Herbert Amann for inviting me to write this book which is based on my Lipschitz lectures in Bonn 1997. I am also grateful to its audience for their interest. The first version of the book was included in Series of Lipschitz Lecture Notes as volume 44 (2002). It was also included in Hokkaido University Technical Report Series in Mathematics as volume 71 (2002). However, since then the author has been fully occupied with the Center of Excellence Programme 'Mathematics of Nonlinear Structures via Singulatiries' (Hokkaido University) and with editing the 4-th version of the Encyclopedic Dictionary of Mathematics. Moreover, the author moved to Tokyo from Sapporo in the middle of 2004. So it has taken a rather long time to complete the present version of this book. After the first version appeared, the field continued to grow and many new articles have been published. They could not all be included without significant expansion of the text. Although some effort was made to cite them, the reference list is not intended to be exhaustive.

I am grateful to Dr. Mi-Ho Giga, Professor Katsuyuki Ishii, Professor Masaki Ohnuma, Dr. Takeshi Ohtsuka, Professor Reiner Schätzle and Professor Kazuyuki Yamauchi for their critical remarks on an earlier version of this book. I am also grateful to Professor Naoyuki Ishimura who read the first version line by line and provided numerous useful suggestions. Without his careful reading it would have been almost impossible to prepare a final version. I am also grateful to anonymous referees for their constructive remarks on the first version. The major part of the book was written when the author was a faculty member of the Hokkaido University. I am grateful to my colleague in Hokkaido University for encouragement. The finacial support of the Japan Society for the Promotion of Science (no. 10304010, 11894003, 12874024, 13894003, 14204011, 15634008, 17654037), the formation of COE 'Mathematics of Nonlinear Structures via Singularities' (Hokkaido University) is gratefully acknowledged. Finally I am grateful to Ms. Hisako Morita (neé Iwai) and Ms. Mika Marubishi for careful typing of respectively, the first and the final version of the manuscripts in latex style.

Y. Giga

Tokyo
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Introduction

In various fields of science there often arise phenomena in which phases (of materials) can coexist without mixing. A surface bounding the two phases is called a *phase boundary*, an *interface* or *front* depending upon the situation. In the process of phase transition a phase boundary moves by thermodynamical driving forces. Since evolution of a phase boundary is unknown and it should be determined as a part of the solution, the problem including such a phase boundary is called in general a free boundary problem. The motion of a phase boundary between ice and water is a typical example, and it has been well studied — the Stefan problem. For classical Stefan problems the reader is referred to the books of L. I. Rubinstein (1971) and of A. M. Meirmanov (1992). The reader is referred to the book of A. Visintin (1996) for free boundary problems related to phase transition. In the Stefan problem, evolution of a phase boundary is affected by the physical situation of the exterior of the surface. However, there is a special but important class of problems where evolution of a phase boundary does not depend on the physical situation outside the phase boundary, but only on its geometry. The equation that describes such motion of the phase boundary is called a *surface evolution equation* or *geometric evolution equations*. There are several examples in material sciences and the equation is also called an *interface controlled model*. Examples are not limited to material sciences. Some of them come from geometry, crystal growth problems and image processing. An important subclass of surface evolution equations consists of equations that arise when the normal velocity of the surface depends locally on its normal and the second fundamental form as well as on position and time. In this book we describe an analytic foundation of the level set method which is useful to analyse such surface evolution equations including the mean curvature flow equation as a typical example. We intend to give a systematic and synthetic approach since the results are scattered in the literature although there are several review articles, in particular a lecture note by L. Ambrosio and N. Dancer (2000). This book also includes several new results on barrier solutions (Chapter 5).

We consider a family $\{\Gamma_t\}_{t\geq 0}$ of hypersurfaces embedded in N-dimensional Euclidean space $\mathbf{R}^N$ parametrized by time t. We assume that Γ_t is a compact hypersurface so that Γ_t is given as a boundary of a bounded open set D_t in $\mathbf{R}^N$

by Jordan–Brouwer's decomposition theorem. Physically, we regard Γ_t as a phase boundary bounding D_t and $\mathbf{R}^N \backslash D_t$ each of which is occupied by different phases. To write down a surface evolution equation we assume that Γ_t is smooth and changes its shape smoothly in time. Let $\mathbf{n}$ be a unit normal vector field of Γ_t outward from D_t. Let $V = V(x,t)$ be the *normal velocity* in the direction of $\mathbf{n}$ at a point of Γ_t. If V depends locally on normal $\mathbf{n}$ and the second fundamental form $-\nabla \mathbf{n}$ of Γ_t, as well as on position x and time t, a general form of surface evolution equation is

$$V = f(x, t, \mathbf{n}, \nabla \mathbf{n}) \quad \text{on } \Gamma_t, \tag{0.0.1}$$

where f is a given function. We list several examples of (0.0.1).

(1) Mean curvature flow equation: $V = H$, where H is the sum of all principal curvatures in the direction of $\mathbf{n}$ and is called the mean curvature throughout in this book (although many authors since Gauss call the average of principal curvatures the mean curvature). The mean curvature is expressed as $H = -\mathrm{div}\ \mathbf{n}$, where div is the surface divergence on Γ_t. This equation was first proposed by W. W. Mullins (1956) to describe motion of grain boundaries in annealing metals.

(2) Gaussian curvature flow equation: $V = K$, where K is the Gaussian curvature of Γ_t, that is, the product of all principal curvatures in the direction of $\mathbf{n}$. For this problem we take $\mathbf{n}$ inward so that a sphere shrinks to a point in a finite time if it evolves by $V = K$. This equation was proposed by W. J. Firey (1974) to describe shapes of rocks on the seashore.

(3) General evolutions of isothermal interface:

$$\beta(\mathbf{n})V = -a\ \mathrm{div}\ \xi(\mathbf{n}) - c(x,t), \tag{0.0.2}$$

where β is a given positive function on a unit sphere S^{N-1} and a is nonnegative constant and c is a given function. The quantity ξ is the *Cahn–Hoffman vector* defined by the gradient of a given nonnegative positively homogeneous function γ of degree 1, i.e., $\xi = \nabla\gamma$ in $\mathbf{R}^N$. In problems of crystal growth we should often consider the anisotropic property of the surface structure of phase boundaries; in one direction the surface is easy to grow, but in the other direction it is difficult to grow. This kind of thing often happens. The equation (0.0.2) includes this effect and was derived by M. E. Gurtin (1988a), (1988b) and by S. B. Angenent and M. E. Gurtin (1989) from the fundamental laws of thermodynamics and the balance of forces. Note that if $\gamma(p) = |p|$ and $\beta(p) \equiv 1$ with $c \equiv 0$, $a \equiv 1$, then (0.0.2) becomes $V = H$. If $a = 0$, the equation (0.0.2) becomes simpler:

$$V = -c/\beta(\mathbf{n}) \quad \text{on } \Gamma_t. \tag{0.0.3}$$

This equation is a kind of Hamilton–Jacobi equation. If $\beta \equiv 1$ and $c < 0$ is a constant, this equation describes the wave front propagation based on Huygens' principle.

(4) Affine curvature flow equations: $V = K^{1/(N+1)}$ or $V = (tK)^{1/(N+1)}$ which were axiomatically derived by L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel (1993) for applications in image processing. The feature of these equations is that they are invariant by affine transform of coordinates. For this problem we take $\mathbf{n}$ inward as for the Gaussian curvature flow equation.

Examples of surface evolution equations are provided by the singular limit of reaction-diffusion equations as many authors have studied. See for example papers of X.-Y. Chen (1991) and X. Chen (1992).

As we summarized above, surface evolution equations are by now very popular among various branches of sciences especially in image processing. The reader is referred to for example, books of G. Sapiro (2001), F. Guichard and J.-M. Morel (2001), F. Cao (2003) and R. Kimmel (2004) for applications of equations in image processing.

A fundamental question of analysis is to construct a unique family $\{\Gamma_t\}_{t\geq 0}$ satisfying (0.0.1) for given initial hypersurface Γ_0 in $\mathbf{R}^N$. In other words it is the question whether there exists a unique solution $\{\Gamma_t\}_{t\geq 0}$ of the initial value problem for (0.0.1) with $\Gamma_t|_{t=0} = \Gamma_0$. This problem is classified as unique existence of a local solution or of a global solution depending on whether one can construct a solution of (0.0.1) in a short time interval or for infinite time. If the equation (0.0.1) is strictly parabolic in a neighborhood of initial hypersurface Γ_0, then there exists a unique local smooth solution $\{\Gamma_t\}$ for given initial data provided that the dependence of variables in f is smooth. It applies to the mean curvature flow equation and its generalization (0.0.2) with $a > 0$ and smooth β and c for general initial data Γ_0 provided that the Frank diagram

$$\text{Frank } \gamma = \{p \in \mathbf{R}^N;\ \gamma(p) \leq 1\} \tag{0.0.4}$$

has a smooth, strictly convex boundary in the sense that all inward principal curvatures are positive. For the Gaussian curvature flow equations and the affine curvature flow equation the equation may not be parabolic for general initial data. It resembles solving the heat equation backward in time, so for general initial data it is not solvable. However, if we restrict ourselves to strictly convex initial surfaces, the problems are strictly parabolic around the initial surfaces and locally uniquely solvable. A standard method to construct a unique local solution is to analyse an equation of a "height" function, where the evolving surface is parametrized by the height (or distance) from the initial surface. See for example a paper by X.-Y. Chen (1991), where he discussed (0.0.2) with $\gamma(p) = |p|,\ \beta \equiv 1,\ a = 1$. The major machinery is the classical parabolic theory in a book of O. A. Ladyženhskaya, V. A. Solonnikov and N. N. Uralćeva (1968) since the equation of a height function is a strictly parabolic equation of second order (around zero height) although it is nonlinear. For (0.0.3) the equation of a height function is of first order so a local smooth solution can be constructed by a method of characteristics. However, as we see later, such a local smooth solution may cease to be smooth in a finite time and singularities may develop even for the mean curvature flow equation where a lot