Progress in Mathematics

Guy David

Singular Sets of Minimizers for the Mumford-Shah Functional

Ferran Sunyer i Balaguer
Award winning monograph

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Foreword

As this book is published, the study of the Mumford-Shah functional is at a curious stage. There was a quite rapid progress a few years ago, in particular with the work of A. Bonnet, and at the same time the most famous question in the subject, the Mumford-Shah conjecture, is still open. Recall that this conjecture says that in dimension 2, the singular set of a reduced minimizer of the functional is locally a C^1 curve, except at a finite number of points.

In this respect, it is probable that some new ideas are needed, but it seems equally likely that many of the recently developed tools will be useful. For instance, it now would seem quite awkward to try to prove the Mumford-Shah conjecture directly, instead of using blow-up limits and working on global minimizers in the plane.

The official goal of this book is to take the optimistic view that we first need to digest the previous progress, and then things will become easier. So we shall try to describe a lot of the available machinery, in the hope that a large part of it will be used, either for the Mumford-Shah problem, or in some other context.

From the author's point of view, the main reason why the Mumford-Shah functional is important is not really its connection with image segmentation, but the fact that it is a very good model for a whole class of problems with free boundaries and a length or surface term, and that there are very interesting mathematical questions connected with it. Hopefully the techniques discovered for the Mumford-Shah functional will be useful somewhere else too, just as some of the most significant improvements in Mumford-Shah Theory come from different areas.

It should probably be stressed at the outset that there is a life outside of the Mumford-Shah conjecture; there are many other interesting (and perhaps easier) questions related to the functional, in particular in dimension 3; we shall try to present a few in the last section of the book.

The project of this book started in a very usual way: the author taught a course in Orsay (fall, 1999), and then trapped himself into writing notes, that eventually reached a monstrous size. What hopefully remains from the initial project is the will to be as accessible and self-contained as possible, and not to treat every aspect of the subject. In particular, there is an obvious hole in our treatment: we shall almost never use or mention the bounded variation approach, even though this approach is very useful, in particular (but not only) with existence results. The author agrees that this attitude of avoiding BV looks a lot like ignoring progress, but he has a good excuse: the BV aspects of the theory have been treated very well in the book of Ambrosio, Fusco, and Pallara [AFP3], and there would be no point in doing the same thing badly here.

Part A will be a general presentation of the Mumford-Shah functional, where we shall discuss its origin in image segmentation, existence and nonuniqueness of minimizers, and give a quick presentation of the Mumford-Shah conjecture and some known results. We shall also give slightly general and complicated (but I claim natural) definitions of almost-minimizers and quasiminimizers. Incidentally, these are slightly different from definitions with the same names in other sources.

Part B reviews simple facts on the Sobolev spaces $W^{1,p}$ that will be needed in our proofs. These include Poincaré estimates, boundary values and traces on planes and spheres, and a corresponding welding lemma. We shall also discuss the existence of functions u that minimize energy integrals $\int_U |\nabla u|^2$ with given boundary values. Thus this part exists mostly for self-containedness.

Part C contains the first regularity results for minimizers and quasiminimizers in dimension 2, i.e., local Ahlfors-regularity, some useful Carleson measure estimates on $|\nabla u|^p$, the projection and concentration properties, and local uniform rectifiability. The proofs are still very close to the original ones.

Part D is a little more original (or at least was when the project started), but is very much inspired by the work of Bonnet. The main point is to use the concentration property of Dal Maso, Morel, and Solimini to prove that limits of minimizers or almost-minimizers are still minimizers or almost-minimizers (actually, with a small additional topological constraint that comes from the normalization of constants). Bonnet did this only in dimension 2, to study blow-up limits of Mumford-Shah minimizers, but his proof is not really hard to adapt. The results of Part D were also proved and published by Maddalena and Solimini [MaSo4] (independently (but before!), and with a different approach to the concentration property).

Part E contains the C^1-regularity almost-everywhere of almost-minimizers, in dimension 2 only. The proof is not really new, but essentially unreleased so far. As usual, the central point is a decay estimate for the normalized energy $\frac{1}{r}\int_{B(x,r)\setminus K} |\nabla u|^2$; in the argument presented here, this decay comes from a variant Bonnet's monotonicity argument. See Sections 47–48.

In Part F we try to give a lot of properties of global minimizers in the plane (these include the blow-up limits of standard Mumford-Shah minimizers). This part is already a slight mixture of recent results (not always given with full detailed proofs) and some mildly new results. The main tools seem to be the results of Section D (because taking limits often simplifies things), Bonnet's monotonicity argument and a variant (which in the best cases play the role of the standard monotonicity result for minimal surfaces), Léger's formula (63.3) (which allows us to compute u given K), and of course some work by hand.

In Part G we return briefly to almost-minimizers in a domain and use Part F to derive a few additional regularity results. We also check that the standard Mumford-Shah conjecture would follow from its counterpart for global minimizers in the plane.

We decided to wait until Part H for a quicker and less complete description of the situation in higher dimensions. For instance, the result of Ambrosio, Fusco, and Pallara on C^1 regularity almost-everywhere is discussed, but not entirely proved.

Part I contains a description of the regularity of K near the boundary, which we decided to keep separate to avoid confusion; however, most of our inside regularity results still hold at the boundary (if our domain Ω is C^1, say). In dimension 2, we even get a good description of K near $\partial\Omega$, as a finite union of C^1 curves that meet $\partial\Omega$ orthogonally.

We conclude with a small isolated section of questions.

This book is too long, and many arguments look alike (after all, we spend most of our time constructing new competitors). Even the author finds it hard to find a given lemma. To try to ameliorate this, some small arguments were repeated a few times (usually in a more and more elliptic way), and a reasonably large index is available. It is hard to recommend a completely linear reading, but probably the last parts are easier to read after the first ones, because similar arguments are done faster by the end.

To make the book look shorter locally, references like (7) will refer to Display (7) in the current section, and (3.7) will refer to (7) in Section 3. The number of the current section is visible in the running title on the top of each page.

Notation

We tried to make reasonable choices; here are just a few:

C is a positive, often large constant that may vary from line to line,

$B(x, r)$ is the open ball centered at x with radius r,

λB is the ball with same center as B, and radius λ times larger,

ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$,

$\widetilde{\omega}_{n-1}$ is the H^{n-1}-measure of the unit sphere in $\mathbb{R}^n$,

$\Box$ signals the end of a proof,

$\subset\subset$ means relatively compact in.

See the index for a few other symbols, that come with a definition.

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A. Presentation of the Mumford-Shah Functional

In this first part we want to give a general presentation of the Mumford-Shah functional. We shall define the functional and rapidly discuss some basic issues like the existence of minimizers, the lack of uniqueness in general, and the fact that the functional becomes much easier to study when the singular set K is fixed. We shall also present the Mumford-Shah conjecture on the regularity of minimizers in dimension 2, and give a few hints on the contents of this book. The part will end with two sections of definitions of almost-minimizers and quasiminimizers, which we think are reasonably important.

1 The Mumford-Shah functional and image segmentation

In this section we want to describe the origin of the Mumford-Shah problem, in connection with the issue of image segmentation. Part of this description is fairly subjective, and this introduction may not reflect much more than the author's personal view on the subject.

Consider a simple domain $\Omega \subset \mathbb{R}^n$. For image segmentation, the most important case is when $n = 2$, and Ω may as well be a rectangle. Also let $g \in L^\infty(\Omega)$ be given. We think of g as a representation of an image; we shall take it real-valued to simplify the exposition, but vector values are possible (for color pictures, or textures, etc.) and would lead to a similar discussion.

The point of image segmentation is to replace g with a simpler function (or image) u which captures "the main features" of g. A very natural idea is to define a functional that measures how well these two contradictory constraints (simplicity and good approximation of g) are satisfied by candidates u, and then minimize the functional.

Of course there are many possible choices of functionals, but it seems that most of those which have been used in practice have the same sort of structure as the Mumford-Shah functional studied below. See [MoSo2] for a thorough description of this and related issues.

The Mumford-Shah functional was introduced, I think, in 1985 [MuSh1], but the main reference is [MuSh2]. The good approximation of g by u will be measured in the simplest way, i.e., by

$$A = \int_{\Omega} |u - g|^2. \tag{1}$$

This is a fairly reasonable choice, at least if we don't have any information a priori on which sort of image g we have. (We may return to this issue soon.) Of course minor modifications, such as using an L^p-norm instead of L^2, or integrating $|u - g|^2$ against some (slowly varying) weight are possible, and they would not really change the mathematics in this book. Note that we do not want to imply here that images are well described by L^2 functions. (L^∞ functions with bounded variation, for instance, would be better, because of the importance of edges.) We just say that for image segmentation we prefer to use a weak norm (like the L^2 norm) in the approximation term. If nothing else, it should make the process less dependent on noise.

Let us now say what we shall mean by a simple function u. We want to authorize u to have singularities (mainly, jumps) along a closed set K, but we want K to be as "simple" as possible. For the Mumford-Shah functional, simple will just mean short: we shall merely measure

$$L = H^{n-1}(K), \tag{2}$$

the Hausdorff measure of dimension $n - 1$ of K (if we work in $\Omega \subset \mathbb{R}^n$). See the next section for a definition; for the moment let us just say that $H^{n-1}(K)$ is the same as the surface measure of K (length when $n = 2$) when K is a reasonably smooth hypersurface (which we do *not* assume here).

The reader may be surprised that we confuse length with simplicity. The objection is perhaps a little less strong when everything is discretized, but also one of the good features of the Mumford-Shah functional is that, for minimizers, K will turn out to have some nontrivial amount of regularity, which cannot be predicted immediately from the formula but definitely makes its choice more reasonable.

Here also many other choices are possible. A minor variant could be to replace L with $\int_K a(x) dH^{n-1}(x)$ for some smooth, positive function a; we shall try to accommodate this variant with some of the definitions of almost-minimizers below. One could also replace H^{n-1} with some roughly equivalent measurement of surface measure, which would not be isotropic but where for instance horizontal and vertical directions would be privileged. This may look more like what happens when one discretizes; in this case we cannot expect K to be C^1 for minimizers, but some of the weaker regularity properties (like local Ahlfors-regularity and uniform rectifiability) will remain true. And this is one of the points of the definition of quasiminimizers in Section 7.

In some cases, one requires K to be more regular directly, by replacing L with the integral over K (against Hausdorff measure) of some function of the curvature of K. We shall not study these variants, but they make sense: after all, if you want K to be fairly smooth, you may as well require this up front instead of trying to get it too indirectly.