Lecture Notes in Applied and Computational Mechanics 69

Koichi Hashiguchi

Elastoplasticity Theory

Second Edition

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Preface

The first edition of this book was published in 2009 to provide the essentials of elastoplasticity for students, engineers and scientists in the field of applied mechanics. Nowadays, the elastoplasticity is developing rapidly responding to highly advancing industry. Then, the finite strain elastoplasticity based on the multiplicative decomposition of the deformation gradient, i.e. the hyperelastic-plastic finite strain theory, which is beyond the level of this book, was published just recently as the book titled "Introduction to Finite Strain Theory for Continuum Elasto-plasticity" by Hashiguchi and Yamakawa (2012), since its foundation has been constructed in recent years. However, it requires further relevant studies to describe wide classes of elastoplastic deformations. The ordinary hypoelastic-based plasticity addressed in the first edition has been highly developed and succeeded to describe the wide classes of elastoplastic/sliding behavior of solids in recent years. Then, the author has made up his mind to publish the second edition, incorporating the novel and relevant theories and elaborating the explanations for readers ranging from beginners to specialists of the elastoplasticity to understand more easily and clearly.

The deformation analysis of solids and structures with high accuracy is required increasingly to enhance their mechanical performance, strength and durability. The basis for the deformation analysis can be provided through elastoplastic deformation analysis amongst others. For that reason, industrial engineers in the fields of mechanical, civil, architectural, aerospace engineering, etc. are required to learn pertinent knowledge of elastoplasticity. In writing this book, standard theories were carefully selected from various formulations and models proposed to date, which are relevant to the steady and efficient study of the elastoplasticity and which will remain universally in the history of elastoplasticity.

Numerous books about elastoplasticity have been published since "Mathematical Theory of Plasticity", the notable book of R. Hill (1950), was written in the middle of the last century. That and similar books mainly address almost only the *conventional elastoplastic constitutive equations*. However, the conventional elastoplastic constitutive equations are formulated on the premise that the interior of yield surface is an elastic domain and thus it is limited to the description of monotonic loading behavior, while the elastoplastic constitutive equations aiming at describing the plastic strain rate induced by the rate of stress inside the yield surface are called the *unconventional constitutive equations* (Drucker, 1988). The unconventional elastoplastic constitutive equation has been studied during a half century since 1960's, aiming at describing general elastoplastic deformation behavior unlimited to the monotonic loading behavior. During that period, a book addressing unconventional elastoplastic constitutive equation has hardly been published so that books on conventional elastoplastic constitutive equation formulated in the olden time have been published repeatedly, since the unconventional elastoplastic constitutive equation was immature. It was quite unsound and strange situation in the development history of elastoplasticity. In recent years, however, the formulation of the unconventional elastoplastic constitutive equation was attained by introducing the concept of subloading surface in the recent years, which is capable of describing cyclic, non-proportional, non-local, rate-dependent deformation of wide classes of materials, e.g. metals and soils and friction phenomena between solids in high accuracy and in high numerical efficiency. This book addresses the unconventional elastoplastic constitutive equation based on the subloading surface concept in detail. It is expected that publication of books on unconventional elastoplastic constitutive equations will be activated after this book.

The author has lectured applied mechanics and has investigated elastoplasticity for nearly a half century, during which time elastoplasticity has made great progress. Various lecture notes, research papers, review articles in English or Japanese, and books in Japanese on these subjects are piled at hand. At present, the author is continuing composition of a monograph on elastoplasticity that has been published serially in a monthly journal from June 2007, to be completed at the end of 2012, while unfortunately it is written in Japanese. Based on those teaching and research materials, this book comprehensively addresses fundamental concepts and formulations of phenomenological elastoplasticity from the conventional to latest theories. Especially, the subloading surface model falling within the framework of the unconventional plasticity model is introduced in detail, which enables us to predict rigorously the plastic strain rate induced by the rate of stress inside the yield surface, comparing it with the other unconventional models. The viscoplastic model is also presented; it is applicable to prediction of deformation behavior in the wide range of strain rate from the quasi-static to the impact loads. Explicit constitutive equations of metals and soils are given for practical application of the theories. In addition, constitutive models of friction of solids based on the concept of the subloading surface are described because they are indispensable for analysis of boundary value problems. Various theories proposed by the author himself are included among the contents in this book in no small number. Their detailed explanations would be possible but, on the other hand, they would easily fall into subjective explanations. For that reason, particular care was devoted to keep the objectivity in the presentation.

The main purpose of this book is to expedite the application of elastoplasticity theory to analyses of engineering problems in practice. Consequently, the salient feature of this book is the exhaustive explanation of elastoplasticity, which is intended to be understood easily and clearly not only by researchers but also by beginners in the field of applied mechanics, without reading any other book. Therefore, mathematics including vector-tensor analysis and the fundamentals in continuum mechanics are first explained to the degree necessary to understand the elastoplasticity theory described in subsequent chapters. For that reason, circumstantial explanations of physical concepts and formulations in elastoplasticity are given without a logical jump such that the derivations and transformations of all equations are described without abbreviation. Besides, general formulations unlimited to a particular material are first addressed in detail since deformations of materials obey common fundamental characteristics, which would provide the universal knowledge for deformation of materials more than describing each formulation for particular materials. Thereafter, explicit constitutive equations of metals, soils and friction phenomena are presented in detail, specifying material functions involved in the general formulation. Without difficulty, readers will be able to incorporate the equations included in this book into their computer programs. The author expects that a wide audience including students, engineers, and researchers of elastoplasticity will read this book and that this work will thereby contribute to the steady development of the study of elastoplasticity and applied mechanics.

As a foundation, the mathematical and the physical ingredients of the continuum mechanics are treated in Chapters 1 to 4. Chapter 1 addresses vector-tensor analysis since physical quantities used in continuum mechanics are tensors; consequently, their relations are described mathematically using tensor equations. Explanations for mathematical properties and rules of tensors are presented to the extent that is sufficient to understand the subject of this book: elastoplasticity theory. Chapter 2 addresses the description of motion and strain (rate) and their related quantities. Chapter 3 presents conservation laws of mass, momentum, and angular momentum, and equilibrium equations and virtual work principles derived from them. In addition, their rate forms used for constitutive equations of inelastic deformation are explained concisely.

Chapter 4 specifically addresses the objectivity of constitutive equations, which is required for the description of material properties. The substantial physical meaning of the objective rate of tensor is explained incorporating the convected base. Then, the objectivities of various stress, strain and their rates are described by examining their coordinate transformation rules. Then, the pull-back and the push-forward operations are systematically explained, defining the Eulerian and the Lagrangian vectors and tensors. Further, all the objective and the corotational time-derivatives of tensors are derived systematically from the convected (embedded) time-derivative. The mathematical proof is given to the fact that the material-time derivative of scalar-valued tensor function can be transformed to the corotational time-derivative of that.

Chapter 5 specifically examines the description of elastic deformation. Elastic constitutive equations are classified into hyperelasticity, Cauchy elasticity and hypoelasticity depending on their levels of reversibility. The mathematical and physical characteristics of these equations are explained prior to the description of elastoplastic constitutive equations in the subsequent chapters.

Elastoplastic constitutive equations are described comprehensively in Chapters 6–9. In Chapter 6, the physical and mathematical backgrounds are first given to the additive decomposition of strain rare (symmetric part of the velocity gradient) into the elastic and the plastic parts and that of the continuum spin (ant-symmetric part of the velocity gradient) into the elastic and the plastic parts based on the multiplicative decomposition of deformation gradient which provides the exact decomposition of deformation gradient tensor into the elastic and the plastic parts by introducing the intermediate configuration as the hyper-elastically unloaded state to the stress free state. In addition, the physical backgrounds are given to facts that the elastic spin designates the sum of the rigid-body rotational rate and the small elastic rotational rate of material substructure and the plastic spin designates the rotational rate of the intermediate configuration. Thereafter, the basic formulations of elastoplastic constitutive equation, e.g. the elastic and the plastic strain rates, the consistency condition, the plastic flow rule and the loading criterion. Descriptions of anisotropy and the tangential inelastic strain rate are also incorporated. However, they fall within the framework of conventional plasticity on the premise that the interior of the yield surface is an elastic domain. Therefore, they are incapable of predicting a smooth transition from the elastic to plastic state and a cyclic loading behavior of real materials pertinently.

In Chapter 7, the continuity and the smoothness conditions are described first. They are the fundamental requirements for the constitutive equations for irreversible deformation, especially to describe cyclic loading behavior accurately. The subloading surface model is described in detail, which falls within the framework of the unconventional plasticity excluding the assumption that the interior of yield surface is an elastic domain. It satisfies both the continuity and the smoothness conditions. In chapter 8, cyclic plasticity models are classified into the models based on the translation of (sub)yield surface(s), i.e. the kinematic hardening and the model based on the expansion/contraction of loading surface. Further, their mathematical structures and mechanical features are explained in detail. It is revealed that the cyclic plasticity models based on the kinematic hardening, e.g. the multi, the two and the superposed nonlinear-kinematic hardening single surface models are the temporizing models, which do not possess a generality/pertinence and contain various serious deficiencies. It is concluded that only the extended subloading surface model falling within the framework of the latter possesses the generality and the mathematical structure capable of describing the cyclic loading behavior of elastoplastic materials, including metals and soils.

In Chapter 9, the formulation of the extended subloading surface model is described in detail. Therein, the inelastic strain rate attributable to the stress rate tangential to the subloading surface is incorporated, which is indispensable for the accurate prediction of non-proportional loading behavior and the plastic instability phenomena. In chapters 10 and 11, constitutive equations based on the subloading surface model are shown for metals and soils. Their validities are verified by the comparisons with various test data containing the cyclic loading.

In chapter 12, the history of the development of the viscoplastic constitutive equation for describing rate-dependent deformation induced in the stress level over the yield surface is reviewed first. Then, the pertinent viscoplastic constitutive equation is described, in which the concept of the subloading surface is incorporated into the overstress model. It is applicable to the prediction of rate-dependent deformation behavior from quasi-static to impact loads, while the deformation behavior under the impact load cannot be described by the past overstress models. On the other hand, it is revealed that the creep model contains impertinence for the description of a quasi-static deformation behavior, although it has been studied widely.

Special issues related to elastoplastic deformation behavior are discussed in Chapters 13 and 14. Chapter 13 specifically examines corotational rate tensors, the necessity of which is suggested in Chapter 4. Mechanical features of corotational tensors with various spins are examined comparing their simple shear deformation characteristics. The pertinence of the plastic spin is particularly explained. Chapter 14 opens with a mechanical interpretation for the localization of deformation inducing a shear band. Then, the approaches to the prediction of shear band inception condition, the inclination/thickness of shear band and the eigenvalue analysis and the gradient theory are explained. The smeared model, i.e., the shear-band embedded model for the practical finite element analysis is also described.

Chapter 15 addresses the prediction of friction phenomena between solid bodies. All bodies except those floating in a vacuum are contacting with other bodies so that the friction phenomena occur between their contact surfaces. Pertinent analyses, not only of the deformation behavior of bodies but also of friction behavior on the contact surface, are necessary for the analyses of boundary-value problems. A constitutive equation of friction is formulated in the similar form to the elastoplastic constitutive equation by incorporating the concept of the subloading surface, which is called the subloading-friction model. It is capable of describing the transition from a static to a kinetic friction attributable to plastic softening and the recovery of the static friction attributable to creep hardening. The anisotropy based on the orthotropy and the rotation of sliding-yield surface is incorporated. The stick-slip phenomenon is analyzed by incorporating the subloading-friction model. Their validities are shown by the comparison with various test data.

The FEM analysis based on elastoplastic constitutive equations described in the former chapters requires pertinent numerical method. In the final chapter 16, the return-mapping and the consistent (algorithmic) tangent modulus tensor are explained in detail, which provides the calculation in a high accuracy and efficiency.

The distinguishable features and importance of this book is the comprehensive descriptions of fundamental concepts and formulations including the objectivity, the objective derivative of tensor function, the associated flow rule, the loading criterion, the continuity and smoothness conditions and their deep physical interpretations in addition to the wide classes of reversible/irreversible constitutive equations for monotonic, cyclic and non-proportional loading behavior, rate-dependent deformation behavior and friction behavior of solids.

The theories described in this book fall within the framework of the hypoelastic-based plasticity for the finite deformation under the infinitesimal elastic deformation. It is recommendable for readers interested in the exact finite elastoplasticity theory to read the companion book "Introduction to Finite Strain Theory for Continuum Elasto-Plasticity" (Hashiguchi and Yamakawa, 2012).

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