COLIN J. BUSHNELL
GUY HENNIART

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THE LOCAL LANGLANDS CONJECTURE FOR GL(2)

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For Elisabeth and Lesley

Foreword

This book gives a complete and self-contained proof of Langlands' conjecture concerning the representations of GL(2) of a non-Archimedean local field. It has been written to be accessible to a doctoral student with a standard grounding in pure mathematics and some extra facility with local fields and representations of finite groups. It had its origins in a lecture course given by the authors at the first Beijing-Zhejiang International Summer School on p-adic methods, held at Zhejiang University Hangzhou in 2004. We hope this is found a fitting response to the efforts of the organizers and the enthusiastic contribution of the student participants.

King's College London and
Université de Paris-Sud.

Contents

Introduction 1
Notation 3
Notes for the reader 4

1 Smooth Representations 7
1. Locally Profinite Groups 8
2. Smooth Representations of Locally Profinite Groups 13
3. Measures and Duality 25
4. The Hecke Algebra 33

2 Finite Fields 43
5. Linear Groups 43
6. Representations of Finite Linear Groups 45

3 Induced Representations of Linear Groups 49
7. Linear Groups over Local Fields 50
8. Representations of the Mirabolic Group 56
9. Jacquet Modules and Induced Representations 61
10. Cuspidal Representations and Coefficients 69
10a. Appendix: Projectivity Theorem 73
11. Intertwining, Compact Induction and Cuspidal Representations 76

4 Cuspidal Representations 85
12. Chain Orders and Fundamental Strata 86
13. Classification of Fundamental Strata 95
14. Strata and the Principal Series 100
15. Classification of Cuspidal Representations 105
16. Intertwining of Simple Strata 111
17. Representations with Iwahori-Fixed Vector 115

5 Parametrization of Tame Cuspidals . . . 123
18. Admissible Pairs . . . 123
19. Construction of Representations . . . 125
20. The Parametrization Theorem . . . 129
21. Tame Intertwining Properties . . . 131
22. A Certain Group Extension . . . 134

6 Functional Equation . . . 137
23. Functional Equation for GL(1) . . . 138
24. Functional Equation for GL(2) . . . 147
25. Cuspidal Local Constants . . . 155
26. Functional Equation for Non-Cuspidal Representations . . . 162
27. Converse Theorem . . . 170

7 Representations of Weil Groups . . . 179
28. Weil Groups and Representations . . . 180
29. Local Class Field Theory . . . 186
30. Existence of the Local Constant . . . 190
31. Deligne Representations . . . 200
32. Relation with ℓ-adic Representations . . . 201

8 The Langlands Correspondence . . . 211
33. The Langlands Correspondence . . . 212
34. The Tame Correspondence . . . 214
35. The ℓ-adic Correspondence . . . 221

9 The Weil Representation . . . 225
36. Whittaker and Kirillov Models . . . 226
37. Manifestation of the Local Constant . . . 230
38. A Metaplectic Representation . . . 236
39. The Weil Representation . . . 245
40. A Partial Correspondence . . . 249

10 Arithmetic of Dyadic Fields . . . 251
41. Imprimitive Representations . . . 251
42. Primitive Representations . . . 257
43. A Converse Theorem . . . 262

11 Ordinary Representations . . . 267
44. Ordinary Representations and Strata . . . 267
45. Exceptional Representations and Strata . . . 279

12 The Dyadic Langlands Correspondence 285
46. Tame Lifting 286
47. Interior Actions 295
48. The Langlands-Deligne Local Constant modulo Roots of Unity 297
49. The Godement-Jacquet Local Constant and Lifting 304
50. The Existence Theorem 307
51. Some Special Cases 313
52. Octahedral Representations 316

13 The Jacquet-Langlands Correspondence 325
53. Division Algebras 326
54. Representations 328
55. Functional Equation 331
56. Jacquet-Langlands Correspondence 334

References 339

Index 345
Some Common Symbols 349
Some Common Abbreviations 351

Introduction

We work with a non-Archimedean local field F which, we always assume, has finite residue field of characteristic p. Thus F is either a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers or a field $\mathbb{F}_{p^r}((t))$ of formal Laurent series, in one variable, over a finite field. The arithmetic of F is encapsulated in the *Weil group* $\mathcal{W}_F$ of F: this is a topological group, closely related to the Galois group of a separable algebraic closure of F, but with rather more sensitive properties. One investigates the arithmetic via the study of continuous (in the appropriate sense) representations of $\mathcal{W}_F$ over various algebraically closed fields of characteristic zero, such as the complex field $\mathbb{C}$ or the algebraic closure $\overline{\mathbb{Q}}_\ell$ of an ℓ-adic number field.

Sticking to the complex case, the one-dimensional representations of $\mathcal{W}_F$ are the same as the characters (i.e., continuous homomorphisms) $F^\times \to \mathbb{C}^\times$: this is the essence of *local class field theory.* The n-dimensional analogue of a character of $F^\times = \mathrm{GL}_1(F)$ is an irreducible smooth representation of the group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices over F. As a specific instance of a wide speculative programme, Langlands [55] proposed, in a precise conjecture, that such representations should parametrize the n-dimensional representations of $\mathcal{W}_F$ in a manner generalizing local class field theory and compatible with parallel global considerations.

The excitement provoked by the local Langlands conjecture, as it came to be known, stimulated a period of intense and widespread activity, reflected in the pages of [8]. The first case, where $n = 2$ and F has characteristic zero, was started in Jacquet-Langlands [46]; many hands contributed but Kutzko, bringing two new ideas to the subject, completed the proof in [52], [53]. Subsequently, the conjecture has been proved in all dimensions, first in positive characteristic by Laumon, Rapoport and Stuhler [58], then in characteristic zero by Harris and Taylor [38], also by Henniart [43] on the basis of an earlier paper of Harris [37].

Throughout the period of this development, the subject has largely remained confined to the research literature. Our aim in this book is to provide a navigable route into the area with a complete and self-contained account of the case $n = 2$, in a tolerable number of pages, relying only on material readily available in standard courses and texts. Apart from a couple of unavoidable caveats concerning Chapter VII, we assume only the standard representation theory for finite groups, the beginnings of the theory of local fields and some very basic notions from topology.

In consequence, our methods are entirely local and elementary. Apart from Chapter I (which could equally serve as the start of a treatise on the representation theory of p-adic reductive groups) and some introductory material in Chapter VII, we eschew all generality. Whenever possible, we exploit special features of GL(2) to abbreviate or simplify the arguments.

The desire to be both compact and complete removes the option of appealing to results derived from harmonic analysis on adèle groups ("base change" [57], [1]) which originally played a determining rôle. This particular constraint has forced us to give the first proof of the conjecture that can claim to be completely local in method.

There is an associated loss, however. The local Langlands Conjecture is just a specific instance of a wide programme, encompassing local and global issues and all connected reductive algebraic groups in one mighty sweep. Beyond the minimal gesture of Chapter XIII, we can give the reader no idea of this. Nor have we mentioned any of the geometric methods currently necessary to prove results in higher dimensions. Fortunately, the published literature contains many fine surveys, from Gelbart's book [32], which still conveys the breadth and excitement of the ideas, to the new directions described in [4].

The approach we take is guided by [46] and [50–53], but we have rearranged matters considerably. We have separated the classification of representations from the functional equation. We have imported ideas of Bernstein and Zelevinsky into the discussion of non-cuspidal representations. While the treatment of cuspidal representations is essentially that of Kutzko, it is heavily informed by hindsight. We have given precedence to the Godement-Jacquet version of the functional equation and so had to treat the Converse Theorem in a novel manner, owing something to ideas of Gérardin and Li. There is also some degree of novelty in our treatment of the Kirillov model and the relation between the functional equation it gives and that of Godement and Jacquet. We have given a quick and explicit proof of the existence of the Langlands correspondence, in the case $p \neq 2$, at an early stage.

The case $p = 2$ has many pages to itself. The method is essentially that of Kutzko, but we have had to bring a new idea to the closing pages (the treatment of the so-called octahedral representations) to avoid an appeal to base change. We regard this case as being particularly important. It remains the one instance of the local conjecture in which the detail is sufficiently complex to be interesting, yet sufficiently visible to illuminate the miracle that is the Langlands correspondence. Even after 25 years, it stands as a sturdy corrective to over-optimistic attitudes to more general problems.

As light relief, we have broadened the picture with some discussion of ℓ-adic representations, since these provide a forum in which the correspondence finds much of its application.

The final Chapter XIII stands outside the main sequence. There, D is the quaternion division algebra over F. The irreducible representations of $D^\times = \mathrm{GL}_1(D)$ can be classified by a method parallel to that used for $\mathrm{GL}_2(F)$. The Jacquet-Langlands correspondence provides a canonical connection between the representation theories of $D^\times$ and $\mathrm{GL}_2(F)$. We include it as an indication of further dimensions in the subject. Given the experience of $\mathrm{GL}_2(F)$, it is a fairly straightforward matter which we have left as a sequence of exercises.

Acknowledgement. The final draft was read by Corinne Blondel, whose acute comments led us to remove a large number of minor errors and obscurities, along with a couple of more significant lapses. It is a pleasure to record our debt to her.

Notation

We list some standard notations which we use repeatedly, without always recalling their meaning.

F = *a non-Archimedean local field;*
$\mathfrak{o}$ = *the discrete valuation ring in* F*;*
$\mathfrak{p}$ = *the maximal ideal of* $\mathfrak{o}$*;*
$\boldsymbol{k}$ = $\mathfrak{o}/\mathfrak{p}$; p = *the characteristic of* $\boldsymbol{k}$*;* $q = |\boldsymbol{k}|$;
U_F = *the group of units of* $\mathfrak{o}$*;* $U_F^n = 1+\mathfrak{p}^n,\ n \geqslant 1$.

(Thus the characteristic of F is 0 or p: we never need to impose any further restriction.) In addition, $\upsilon_F : F^\times \to \mathbb{Z}$ is the normalized (surjective) additive valuation and $\|x\| = q^{-\upsilon_F(x)}$. We denote by $\boldsymbol{\mu}_F$ the group of roots of unity in F of order prime to p.

If E/F is a finite field extension, we use the analogous notations $\mathfrak{o}_E$, $\mathfrak{p}_E$, etc. The norm map $E^\times \to F^\times$ is denoted $\mathrm{N}_{E/F}$, and the trace $E \to F$ is $\mathrm{Tr}_{E/F}$. The ramification index and the residue class degree are $e(E|F)$, $f(E|F)$ respectively. The discriminant is $\mathfrak{d}_{E/F} = \mathfrak{p}^{d+1}$, $d = d(E|F)$.

The symbol tr is reserved for the trace of an endomorphism, such as a matrix or a group representation, and det is invariably the determinant.

If R is a ring with 1, $R^\times$ is its group of units and $\mathrm{M}_n(R)$ is the ring of $n\times n$ matrices over R. When R is commutative, $\mathrm{GL}_n(R)$ (resp. $\mathrm{SL}_n(R)$) is the group of $n \times n$ matrices over R which are invertible (resp. of determinant 1). We use the notation B, T, N, Z for the subgroups of GL_2 of matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

respectively. Unless otherwise specified, $A = \mathrm{M}_2(F)$ and $G = \mathrm{GL}_2(F)$.

Notes for the reader

Prerequisites. We assume the beginnings of the representation theory of finite groups, including Mackey theory: the first 11 sections of [77] cover it all, bar a couple of results requiring reference to [26]. Of non-Archimedean local fields, we need general structure theory as far as the discriminant and structure of tame extensions, plus behaviour of the norm in tame or quadratic extensions. Practically everything can be found in [30] or the first two parts of [76], while [87] is the source of many of the ideas here. From topology and measure theory, beyond the most elementary concepts, we cover practically everything we need.

All this material is commonly available in many books: we mention only personal favourites.

From Chapter VII onwards, we rely on local class field theory. No detail is involved, so we have been able to take an axiomatic approach. The reader might consult the compact [68] or [74], [76]. More serious is the treatment in §30 of the existence of the Langlands-Deligne local constant. This depends on an interplay between local and global fields using some deep (but classical) theorems. The reader could again take an axiomatic approach. We have included a brief account which is complete modulo the classical background. (The requisite material is in [68] or [54].)

Navigation. Sections are numbered consecutively throughout the book. Each section is divided into (usually) short paragraphs, numbered in the form y.z. A reference y.z Proposition means the (only) proposition in paragraph y.z.

Chapter I stands alone, and could serve as an introduction to much wider areas. Chapter II is elementary, and could be read first. Parts of Chapters VII and X can be read independently. Chapter XIII could be read directly after Chapter VI. Otherwise, the logical dependence is linear and fairly rigid.

Principal series (or non-cuspidal) representations form a distinct subtheme. At a first reading, this could be edited out or pursued exclusively, according to taste. (For a different approach, emphasizing non-cuspidal representations and their importance for L-functions, see Bump's book [10].) Another "short course" option would be to stop at the end of Chapter VIII, by which stage the argument is complete for all but dyadic fields F.

Exercises. A few exercises are scattered through the text. These are intended to illuminate, entertain, or to indicate directions we do not follow. Only the simpler ones ever make a serious contribution to the main argument.

Notes. We have appended brief notes or comments to some chapters, to indicate further reading or wider perspectives. They tend to pre-suppose greater experience than the main text.

History. We have written an account of the subject, not its history: that would be a separate project of comparable scope. We have made no attempt at a complete bibliography. We have cited sources of major importance, and those we have found helpful in the preparation of this volume. We have also mentioned a number of recent works, along with older ones that, in our opinion, remain valuable to one learning the subject.