Lecture Notes in Applied and Computational Mechanics 76

Kazumi Watanabe

Integral Transform Techniques for Green's Function

Second Edition

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Dedicated to my teachers,
Dr. Akira Atsumi (Late Professor, Tohoku University)

and

Dr. Kyujiro Kino (Late Professor, Osaka Institute of Technology)

Preface to the Second Edition

After publishing the original version, the author noticed that more detailed mathematical techniques should be included so that the young reader could learn the traditional analytical techniques without any mathematical skip. That is, the discussion on square root functions. In many dynamic/wave problems, we frequently encounter the square root function which is the typical multi-valued function and have to introduce branch cuts in the complex plane for the inversion integral. A simple elementary technique for the introduction of the branch cut and the discussion on the argument of the square root function along the cut are included in Chap. 1 as Sect. 1.3. This branch cut is employed throughout the book and applied to the inversion integrals in Sect. 2.5 and other sections. Due to the introduction of the unified branch cut, Sect. 2.5 for the time-harmonic Green's function is wholly rewritten.

In the revising process the author also noticed five exact closed-form solutions: three are Green's functions for torsion problems, the fourth one is for the reflection problem and the last is for the scattering problem. Green's functions for the torsion problem are inserted in Chaps. 3 and 7. SH-wave reflection at a moving boundary in Sect. 7.4 is slightly rewritten in order to include the closed-form solution. Section 7.5 is newly inserted and shows the exact closed-form solution for a wave scattering problem in an inhomogeneous elastic solid. Further, employing the branch cut described in Chap. 1, an excellent application technique of the complex integral is explained in the last Sect. 7.6. It is the transformation of a semi-infinite integral to a finite one that is suitable for numerical computations. Needless to say, many errors and mistakes in the original version have also been corrected.

The author hopes the young reader can learn one of the traditional analytical techniques, especially the application of the complex integral for the integral transform. Thus, the present revised version is more instructive than the original one, and every question and inquiry via email "kazy@yz.yamagata-u.ac.jp" is welcome.

Hikoshima Island, Japan, January 2015 Kazumi Watanabe

Preface

When I was a senior student, I found a book on the desk of my advisor professor and asked him how to get it. His answer was negative, saying its content was too hard, even for a senior student. Some weeks later, I found it again in a book store, the biggest one in Osaka. This was my first encounter with “Fourier Transforms” written by the late Prof. I. N. Sneddon. Since then, I have learned the power of integral transform, i.e. the principle of superposition.

All phenomena, regardless of their fields of event, can be described by differential equations. The solution of the differential equation contains the crucial information to understand the essential feature of the phenomena. Unfortunately, we cannot solve every differential equation, and almost all phenomena are governed by nonlinear differential equations, of which most are not tractable. The differential equations that can be solved analytically are limited to a very small number. But their solutions give us the essence of the event. The typical partial differential equations that can be solved exactly are the Laplace, the diffusion and the wave equations. These three partial differential equations, which are linearized for simplicity, govern many basic phenomena in physical, chemical and social events. In addition to single differential equations, some coupled linear partial differential equations, which govern somewhat complicated phenomena, are also solvable and their solutions give much information about, for example, the deformation of solid media, propagation of seismic and acoustic waves, and fluid flows.

In a case where phenomena are described by linear differential equations, the solutions can be expressed by superposition of basic/fundamental solutions. The integral transform technique is a typical superposition technique. The integral transform technique does not require any previous knowledge for solving differential equations. It simply transforms partial or ordinary differential equations to reduced ordinary differential equations or to simple algebraic equations. However, a substantial difficulty is present regarding the inversion process. Many inversion integrals are tabulated in various formula books, but typically, this is not enough. If a suitable integration formula cannot be found, the complex integral must be considered and Cauchy’s integral theorem is applied to the inversion integral. Thus, integral transform techniques are intrinsically connected with the theory of complex integrals.

The present book intends to show how to apply integral transforms to partial differential equations and how to invert the transformed solution into the actual space-time domain. Not only the use of integration formula tabulated in books, but also the application of Cauchy's integral theorem for the inversion integrals are described concisely and in detail. A particular solution for a differential equation with a nonhomogeneous term of a point source is called the "Green's function." The Green's functions for coupled differential equations are called "Green's dyadic." The Green's function and Green's dyadic are the basic and fundamental solution of the differential equation and give the principal features of the event. Furthermore, these Green's functions and dyadics have many applications for numerical computation techniques such as the Boundary Element Method. However, the Green's function and Green's dyadic have been scattered in many branches of applied mechanics and thus, their solution methods are not unified. This book intends to present and illustrate a unified solution method, namely the method of integral transform for the Green's function and Green's dyadic. Thus, the fundamental Green's function for the Laplace and wave equations and the Green's dyadic for elasticity equations are gathered in this single book so that the reader can have access to a proper Green's function and understand the mathematical process for its derivation.

Chapter 1 describes roughly the definition of the integral transforms and the distributions to be used throughout the book. Chapter 2 shows how to apply an integral transform for solving a single partial differential equation such as the Laplace equation and the wave equation. The basic technique of the integral transform method is demonstrated. Especially, in the case of the time-harmonic response for the wave equation, the integration path for the inversion integral is discussed in detail. At the end of the chapter, the obtained Green's functions are listed in a table so that the reader can easily find the difference of the functional form among the Green's functions. An evaluation technique for a singular inversion integral which arises in a 2D static problem of Laplace equation is also developed.

The Green's dyadic for 2D and 3D elastodynamic problems are discussed in Chap. 3. Three basic responses, impulsive, time-harmonic and static responses, are obtained by the integral transform method. The time-harmonic response is derived by the convolution integral of the impulsive response without solving the differential equations for the time-harmonic source.

Chapter 4 presents the governing equations for acoustic waves in a viscous fluid. Introducing a small parameter, the nonlinear field equations are linearized and reduced to a single partial differential equation for velocity potential or pressure deviation. The Green's function which gives the acoustic field in a uniform flow is derived by the method of integral transform. A conversion technique for the inversion integral is demonstrated. That is, to transform an inversion integral along the complex line to that along the real axis in the complex plane. It enabled us to apply the tabulated integration formula.

Chapter 5 presents Green's functions for beams and plates. The dynamic response produced by a point load on the surface of a beam and a plate is discussed. The impulsive and time-harmonic responses are derived by the integral transform method. In addition to the tabulated integration formulas, the inversion integrals are evaluated by application of complex integral theory.

Chapter 6 presents a powerful inversion technique for transient problems of elastodynamics, namely the Cagniard-de Hoop method. Transient response of an elastic half space to a point impulsive load is discussed by the integral transform method. Applying Cauchy's complex integral theorem, the Fourier inversion integral is converted to an integral of the Laplace transform and then its Laplace inversion is carried out by inspection without using any integration formula. The Green's function for an SH-wave and Green's dyadics for P, SV and SH-waves are obtained.

The last Chap. 7 presents three special Green's functions/dyadics. The 2D static Green's dyadic for an orthotropic elastic solid and that for an inhomogeneous solid are derived. In the last section, a moving boundary problems is discussed. Two different Laplace transforms are applied for a single problem, and a conversion formula between two Laplace transforms is developed with use of Cauchy's theorem. This conversion enables us to apply the integral transform technique to a moving boundary problem.

The integral transform technique has been used for many years. The inversion process inevitably requires a working knowledge of the theory of complex functions. The author finds the challenge of a complex integral amusing, especially the challenge of choosing the right contour for the inversion integral. He hopes that young researchers will join the fun and carry on with the inversion techniques. In this respect it must be mentioned that he feels a lack of mathematical skill in the recent research activities, since some researchers tend to use numerical techniques without considering the possibility of an analytical solution. The more mathematical techniques expand the horizon of the differential equations wider and one can extract more firm knowledge from the nature which is described by the differential equations. The author hopes that the present book gives one more technique to the younger researchers.

Finally, the author wishes to express his sincere thanks to Dr. Mikael A. Langthjem, Associate Professor of Yamagata University, for his advice and nice comments.

Yonezawa, Japan, January 2013 Kazumi Watanabe

Contents

Chapter 1
Definition of Integral Transforms and Distributions

This first chapter describes a brief definition of integral transforms, such as Laplace and Fourier transforms, a rough definition of delta and step functions which are frequently used as the source function, and a concise introduction of the branch cut for a multi-valued square root function. The multiple integral transforms and their notations are also explained. The introduction of the branch cut and the discussion on the argument of the root function along the branch cut will be helpful for applying the complex integral to the inverse transform. The last short comment lists some important formula books which are crucial for the inverse transform, i.e. the evaluation of the inversion integral.

1.1 Integral Transforms

For a well-defined function $f(x)$, $x \in (a, b)$, when the integral with the kernel function $K(\xi, x)$,

$$F(\xi) = \int_a^b K(\xi, x) f(x) dx \tag{1.1.1}$$

has its inverse integral with another kernel function $K^*(\xi, x)$,

$$f(x) = \int_L K^*(\xi, x) F(\xi) d\xi \tag{1.1.2}$$

we call this integration pair an "integral transform." The function $f(x)$ is an original function and the function $F(\xi)$ is the "image or transformed function" in the transformed domain. If the reciprocal $f(x) \Leftrightarrow F(\xi)$ holds, we call $F(\xi)$ the "integral

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transform” of $f(x)$ and the two-variable-functions $K(\xi, x)$ and $K^*(\xi, x)$ the kernels of the integral transform.

We have already learned many integral transforms which are classified and named depending on the kernel function and the integration range. A well-known integral transform is the Laplace transform. The (one-sided) Laplace transform is, in the present book, defined for the time-variable function $f(t)$, $t \in [0, \infty)$ as

$$f^*(s) = \int_0^\infty f(t) \exp(-st) dt \tag{1.1.3}$$

where “s” is the transform parameter and the transform kernel is $\exp(-st)$. The inverse transform is also defined by the integral along the complex line,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) \exp(st) ds \tag{1.1.4}$$

where the integration path from $c - i\infty$ to $c + i\infty$ is called the “Bromwich line.” The real constant “c” must be larger than the real part of any singular point of the transformed function $f^*(s)$. Thus this line is placed at far right from all singular points in the complex s-plane.

In these definitions, the variable s is called the transform parameter and the two kernels for the transform and the inverse transform are exponential functions:

$$K(s,t) = \exp(-st), \quad K^*(s,t) = \frac{1}{2\pi i} \exp(st) \tag{1.1.5}$$

Further, the integration ranges are also different from each other. The transform integral is carried out along the semi-infinite real line $[0, \infty)$ for the time and the inverse transform is carried out along an infinite line $(c - i\infty, c + i\infty)$ in the complex s-plane.

Since any notation for the transform parameter is available, one should be aware of the notation of the parameter since some authors use “p” instead of “s.” When the Laplace inversion is carried out by using some inversion formulas in a reference book, not performing the inversion integral in the complex plane, the symbolic form of the Laplace inversion

$$f(t) = L^{-1}[f^*(s)] \tag{1.1.6}$$

is used for the sake of simplicity. The present book also employs frequently this simple expression for the Laplace inversion.

So far, many integral transform pairs have been found and defined. We choose and use one suitable integral transform depending on the geometry (integration range) and the simplicity of its application. The followings are typical integral transforms which are much used in applications.