Progress in Mathematical Physics
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Philippe Blanchard
Erwin Brüning

Mathematical Methods in Physics

Distributions, Hilbert Space Operators, Variational Methods, and Applications in Quantum Physics

Second Edition

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ISSN 1544-9998 ISSN 2197-1846 (electronic)
Progress in Mathematical Physics
ISBN 978-3-319-14044-5 ISBN 978-3-319-14045-2 (eBook)
DOI 10.1007/978-3-319-14045-2

Library of Congress Control Number: 2015931210

Mathematics Subject Classification (MSC): 46-01, 46C05, 46F05, 46N50, 47A05, 47L90, 49-01, 60E05, 81Q10

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Dedicated to the memory of
Yurko Vladimir Glaser and Res Jost,
mentors and friends

Preface to the Second Edition

The first edition of this book was published in 2003. Let us first thank everyone who, over the past 11 years, has provided us with suggestions and corrections for improving this first edition. We are extremely grateful for this help.

The past decade has brought many changes, but the aim of this book remains the same. It is intended for graduate students in physics and mathematics and it may also be useful for theoretical physicists in research and in industry. As the extended title of this second edition indicates, we have focused our attention to a large extent on topical applications to Quantum Physics.

With the hope that this book would be a useful reference for people applying mathematics in their work, we have emphasized the results that are important for various applications in the areas indicated above. This book is essentially self-contained. Perhaps some readers will use this book as a compendium of results; this would be a pity, however, because proofs are often as important as results. Mathematical physics is not a passive activity, and therefore the book contains more than 220 exercises to challenge readers and to facilitate their understanding.

This second edition differs from the first through the reorganization of certain material and the addition of five new chapters which have a new range of substantial applications.

The first addition is Chap. 13 "Sobolev spaces" which offers a brief introduction to the basic theory of these spaces and thus prepares their use in the study of linear and nonlinear partial differential operators and, in particular, in the third part of this book dedicated to "Variational Methods."

While in the first edition Hilbert–Schmidt and trace class operators were only discussed briefly in Chap. 22 "Special classes of bounded operators," this edition contains a new Chap. 26 "Hilbert–Schmidt and trace class operators." The remainder of Chap. 22 has been merged with the old, shorter Chap. 23 "Self-adjoint Hamilton operators" to form the new Chap. 23 "Special classes of linear operators," which now also contains a brief section on von Neumann's beautiful application of the spectral theory for unitary operators in ergodic theory.

Chapter 26 "Hilbert–Schmidt and trace class operators" gives a fairly comprehensive introduction to the theory of these operators. Furthermore, the dual spaces of the spaces of compact and of trace class operators are determined, which allows a thorough discussion of several locally convex topologies on the space $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a separable Hilbert space $\mathcal{H}$. These are used later in the chapter "Operator algebras and positive mappings" to characterize normal states on von Neumann algebras. This chapter also contains the definition of the partial trace of trace class operators on the tensor products of two separable infinite dimensional Hilbert spaces and studies its main properties. These results are of particular importance in the theory of open quantum systems and the theory of decoherence.

The motivation for the new Chap. 29 "Spectral analysis in rigged Hilbert spaces" comes from the fact that, on one side, Dirac's bra and ket formalism is widely and successfully employed in theoretical physics, but its mathematical foundation is not easily accessible. This chapter offers a nearly self-contained mathematical basis for this formalism and proves in particular the completeness? there is a word missing in this sentence completeness of the set of generalized eigenfunctions.

In Chap. 30 "Operator algebras and positive mappings," we study in detail positive and completely positive mappings on the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ respectively on its subalgebras. We explain the GNS construction for positive linear functionals in detail and characterize states, i.e., normalized positive linear functionals, in terms of their equivalent continuity properties (normal, completely additive, tracial). Next, Stinespring's factorization theorem characterizes completely positive maps in terms of representations. Since all representations of $\mathcal{B}(\mathcal{H})$ are determined too, we can give a self-contained characterization of all completely positive mappings on $\mathcal{B}(\mathcal{H})$.

The last new chapter, Chap. 31 "Positive mappings in quantum physics," presents several results which are very important for the foundations of quantum physics and quantum information theory. We start with a detailed discussion of Gleason's theorem on the general form of countable additive probability measures on the set of projections of a separable Hilbert space. Using some of the results of the previous chapter, we then give a self-contained characterization of quantum operations specifically, quantum channel maps (Kraus form) and conclude with a brief discussion of the stronger form of these results if the underlying Hilbert space is finite dimensional (Choi's characterization of quantum operations).

On the basis of the mathematical results obtained in this and earlier chapters, it is straightforward to introduce some quite prominent concepts in quantum physics, namely open quantum systems, reduced dynamics, and decoherence. This is done in the last section of this chapter.

Bielefeld and Durban *Ph. Blanchard*
May 2014 *E. Brüning*

Preface

Courses in modern theoretical physics have to assume some basic knowledge of the theory of generalized functions (in particular distributions) and of the theory of linear operators in Hilbert spaces. Accordingly, the faculty of physics of the University of Bielefeld offered a compulsory course *Mathematische Methoden der Physik* for students in the second semester of the second year, which now has been given for many years. This course has been offered by the authors over a period of about 10 years. The main goal of this course is to provide basic mathematical knowledge and skills as they are needed for modern courses in quantum mechanics, relativistic quantum field theory, and related areas. The regular repetitions of the course allowed, on the one hand, testing of a number of variations of the material and, on the other hand, the form of the presentation. From this course, the book *Distributionen und Hilbertraumoperatoren. Mathematische Methoden der Physik. Springer-Verlag Wien, 1993* emerged. The present book is a translated, considerably revised, and extended version of this book. It contains much more than this course since we added many detailed proofs, many examples, and exercises as well as hints linking the mathematical concepts or results to the relevant physical concepts or theories.

This book addresses students of physics who are interested in a conceptually and mathematically clear and precise understanding of physical problems, and it addresses students of mathematics who want to learn about physics as a source and as an area of application of mathematical theories, i.e., all those students with interest in the fascinating interaction between physics and mathematics.

It is assumed that the reader has a solid background in analysis and linear algebra (in Bielefeld this means three semesters of analysis and two of linear algebra). On this basis the book starts in Part A with an introduction to basic linear functional analysis as needed for the Schwartz theory of distributions and continues in Part B with the particularities of Hilbert spaces and the core aspects of the theory of linear operators in Hilbert spaces. Part C develops the basic mathematical foundations for modern computations of the ground state energies and charge densities in atoms and molecules, i.e., basic aspects of the direct methods of the calculus of variations including constrained minimization. A powerful strategy for solving linear and non-linear boundary and eigenvalue problems, which covers the Dirichlet problem and its nonlinear generalizations, is presented as well. An appendix gives detailed proofs of the fundamental principles and results of functional analysis to the extent they are needed in our context.

With great pleasure we would like to thank all those colleagues and friends who have contributed to this book through their advice and comments, in particular G. Bolz, J. Loviscach, G. Roepstorff, and J. Stubbe. Last but not least we thank the editorial team of Birkhäuser—Boston for their professional work.

Bielefeld and Durban
June 2002

Ph. Blanchard
E. Brüning

Acknowledgements

It is with great pleasure that we thank our colleague and friend Shigeaki Nagamachi (Tokushima, Japan) for his help with the proof reading of the new chapters and a number of suggestions for improving our presentation. Our thanks go also to our colleagues Florian Scheck and Ludwig Streit and to many students. They helped us to eliminate quite a number of (small) errors.

Last, but not least, we thank the editorial team of Birkhäuser for their professional work.

Bielefeld and Durban
May 2014

Ph. Blanchard
E. Brüning

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