

Jean-François Ganghoffer
Ivailo Mladenov *Editors*

Similarity and Symmetry Methods

Applications in Elasticity and Mechanics
of Materials

Jean-François Ganghoffer
Ivailo Mladenov *Editors*

Similarity and Symmetry Methods

Applications in Elasticity and Mechanics
of Materials

Lecture Notes in Applied and Computational Mechanics

Volume 73

Series editors

Friedrich Pfeiffer, Technische Universität München, Garching, Germany
e-mail: pfeiffer@amm.mw.tum.de

Peter Wriggers, Universität Hannover, Hannover, Germany
e-mail: wriggers@ikm.uni-hannover.de

For further volumes:
<http://www.springer.com/series/4623>

About this Series

This series aims to report new developments in applied and computational mechanics—quickly, informally and at a high level. This includes the fields of fluid, solid and structural mechanics, dynamics and control, and related disciplines. The applied methods can be of analytical, numerical and computational nature.

Jean-François Ganghoffer · Ivaïlo Mladenov
Editors

Similarity and Symmetry Methods

Applications in Elasticity and Mechanics
of Materials

Editors

Jean-François Ganghoffer
LEMTA—Université de Lorraine
Vandoeuvre
France

Ivaïlo Mladenov
Institute of Biophysics and Biomedical
Engineering
Bulgarian Academy of Sciences
Sofia
Bulgaria

ISSN 1613-7736

ISSN 1860-0816 (electronic)

ISBN 978-3-319-08295-0

ISBN 978-3-319-08296-7 (eBook)

DOI 10.1007/978-3-319-08296-7

Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014944340

© Springer International Publishing Switzerland 2014

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law. The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

On June 06–09, 2013, the EUROMECH Workshop *Similarity, Symmetry and Group Theoretical Methods in Mechanics* took place in Varna, Bulgaria.

It brought together many scientists from European countries and USA, and focused on the current state of the art in the field of similarity methods in mechanics. The aim of this Workshop was to bring together researchers who apply similarity and symmetry analysis to theoretical and engineering problems in both solid and fluid mechanics, researchers who are developing significant extensions of these methods so that they can be applied more widely, and numerical analysts who develop and use such methods in numerical schemes.

The scientific program of the Workshop was built around main speakers who gave an overview of the field in the form of short lecture courses delivered by

Nail H. Ibragimov—Group Analysis as a Microscope of Mathematical Modeling,

George Bluman—Some Recent Developments in Finding Systematically Conservation Laws and Nonlocal Symmetries for Partial Differential Equations, and

Charles-Michel Marle—Symmetries of Hamiltonian Dynamical Systems, Momentum Maps and Reduction.

The two organizers are deeply grateful to EUROMECH for the provided support making possible the first in this new series of scientific meetings. This Springer volume contains lecture notes written by the principal speakers of the Workshop which are complemented by a few shorter contributions dealing with specific problems.

The Editors hope very much that this volume gives a modern overview of the similarity and symmetry methods and shows applications of this active field of research in mechanics and will serve as a reference in the years to come.

Nancy, April 2014
Sofia

Jean-François Ganghoffer
Ivailo Mladenov

Contents

| | |
|---|------------|
| Some Recent Developments in Finding Systematically Conservation Laws and Nonlocal Symmetries for Partial Differential Equations | 1 |
| George Bluman and Zhengzheng Yang | |
| Construction of Conservation Laws Using Symmetries | 61 |
| Nail H. Ibragimov | |
| Symbolic Computation of Nonlocal Symmetries and Nonlocal Conservation Laws of Partial Differential Equations Using the GeM Package for Maple | 165 |
| Alexei F. Cheviakov | |
| Symmetries of Hamiltonian Systems on Symplectic and Poisson Manifolds | 185 |
| Charles-Michel Marle | |
| Symmetries in Mechanics: From Field Theories to Master Responses in the Constitutive Modeling of Materials | 271 |
| Jean-François Ganghoffer | |
| Symmetries and Some Special Solutions of the Helfrich Model | 353 |
| Vladimir I. Pulov, Mariana Ts. Hadzhilazova and Ivaïlo M. Mladenov | |
| Lie Group Analysis of the Willmore and Membrane Shape Equations | 365 |
| Vassil M. Vassilev, Petar A. Djondjorov and Ivaïlo M. Mladenov | |

Some Recent Developments in Finding Systematically Conservation Laws and Nonlocal Symmetries for Partial Differential Equations

George Bluman and Zhengzheng Yang

Abstract This chapter presents recent developments in finding systematically conservation laws and nonlocal symmetries for partial differential equations. There is a review of local symmetries, including Lie's algorithm to find local symmetries in evolutionary form and their applications. The Direct Method for finding local conservation laws is reviewed and its relationship to and extension of Noether's theorem are discussed. Moreover, it is shown how symmetries, including discrete symmetries may yield additional conservation laws from known conservation laws. Systematic procedures are presented to seek nonlocally related PDE systems for a given PDE system with two independent variables. In particular, these procedures include the use of conservation laws, point symmetries, and subsystems (including subsystems arising after appropriate invertible transformations of variables) to obtain trees of equivalent nonlocally related PDE systems. In turn, it is shown how the calculation of point symmetries of such nonlocally related systems leads to the discovery of nonlocal symmetries for a given PDE system. The situation of systematically constructing useful nonlocally related systems in multidimensions is considered. Many illustrative examples are provided.

1 Introduction

This chapter is concerned with recent developments in finding conservation laws (CLs) and nonlocal symmetries for partial differential equations (PDEs). It focuses on recent research of the authors and some of the first author's collaborators, including Stephen Anco, Alexei Cheviakov, Temuer Chaolu, Jean-François Ganghoffer, Nataliya

G. Bluman (✉) · Z. Yang
Department of Mathematics, The University of British Columbia,
Vancouver V6T 1Z2, Canada
e-mail: bluman@math.ubc.ca

Ivanova, Sukeyuki Kumei, Ian Lisle, Alex Ma, Greg Reid, Vladimir Shtelen and Thomas Wolf. Much of the material in this chapter appears in more detail in [1, 2].

In the latter part of the 19th century, Sophus Lie initiated his studies on continuous groups of transformations (Lie groups of transformations) in order to put order to, and thereby extend systematically, the hodgepodge of heuristic techniques for solving ordinary differential equations (ODEs). In particular, Lie showed the following.

- The problem of finding a Lie group of point transformations leaving invariant a differential equation (*point symmetry* of a differential equation) is systematic and reduces to solving a related linear system of determining equations for the coefficients (infinitesimals) of its *infinitesimal generator*.
- A point symmetry of an ODE leads to reducing systematically the order of an ODE (irrespective of any imposed initial conditions).
- A point symmetry of a PDE leads to finding systematically special solutions called *invariant (similarity) solutions*.
- A point symmetry of a differential equation generates a one-parameter family of solutions from any known solution of the differential equation that is not an invariant solution.

However there were limitations to the applicability of Lie's work.

- There were a restricted number of applications for point symmetries, especially for PDE systems.
- Few differential equations have point symmetries.
- For PDE systems having point symmetries, the invariant solutions arising from point symmetries normally yield only a small submanifold of the solution manifold of the PDE system and hence few posed boundary value problems can be solved.
- There was the computational difficulty of finding point symmetries.

Since the end of the 19th century there have been significant extensions of Lie's work on symmetries of PDEs to extend its range of applicability.

- Further applications of point symmetries have been found to include linearizations, other mappings and solutions of boundary value problems. In particular, knowledge of the point symmetries of a nonlinear PDE system (contact symmetries in the case of a scalar PDE), allows one to determine whether the system can be mapped invertibly to a linear system and yields a procedure to find such a mapping when one exists [2–4]. Knowledge of the point symmetries of a linear PDE system with variable coefficients allows one to determine whether the system can be mapped invertibly to a linear system with constant coefficients and yields a procedure to find such a mapping when one exists [2, 3].
- Extensions of the spaces of symmetries of a given PDE system to include local symmetries (higher-order symmetries) as well as nonlocal symmetries [2, 5–8].
- Extension of the applications of symmetries to include variational symmetries that yield conservation laws for variational systems [2, 8].

- Extension of variational symmetries to more general multipliers and resulting conservation laws for essentially any given PDE system [2, 8–11].
- The discovery of further solutions that arise from the extension of Lie’s method to the “nonclassical method” as well as other generalizations [2, 12, 13].
- The development of symbolic computation software to solve efficiently the (overdetermined) linear system of symmetry and/or multiplier determining equations as well as related calculations for solving the nonlinear systems of determining equations arising when one uses the nonclassical method [14–18].

1.1 What is a Symmetry of a PDE System and How to Find One?

A symmetry (discrete or continuous) of a PDE system is any transformation of its solution manifold into itself, i.e., a symmetry transforms (maps) any solution of a PDE system to a solution of the same system. In particular, continuous symmetries of a PDE system are continuous deformations of its solutions to solutions of the same PDE system. Hence continuous symmetries are defined topologically and not restricted to just point or local symmetries. Thus, in principle, any nontrivial PDE has symmetries. The problem is to find and use the symmetries of a given PDE system. Practically, to find symmetries of a given PDE system, one considers transformations, acting locally on the variables of some finite-dimensional space, which leave invariant the solution manifold of the PDE system and its differential consequences. However, these variables do not have to be restricted to just the independent and dependent variables of the given PDE system.

Higher-order symmetries (local symmetries) arise when the solutions of the linear determining equations for infinitesimals are allowed to depend on a finite number of derivatives of dependent variables of the PDE system.

- Infinitesimals for a point symmetry in evolutionary form allow at most linear dependence on first derivatives of dependent variables of a PDE system.
- Infinitesimals for a contact symmetry in evolutionary form (only exists for a scalar PDE) allow arbitrary dependence on at most first derivatives of the dependent variable of a scalar PDE.

In making the extension from point and contact symmetries to higher-order symmetries, it is essential to realize that the linear determining equations for local symmetries are the linearized system (*Fréchet derivative*) of the given PDE system that holds for all of its solutions. Globally, point and contact symmetries act on finite-dimensional spaces whereas higher-order symmetries act on infinite-dimensional spaces consisting of the dependent and independent variables of a given PDE system as well as all of their derivatives. Well-known integrable equations of mathematical physics such as the Korteweg-de-Vries equation have an infinite number of higher-order symmetries [19].

Another extension is to consider solutions of the determining equations where infinitesimals have an ad-hoc dependence on nonlocal variables such as integrals of

the dependent variables [20–23]. For some PDEs, such *nonlocal symmetries* can be found formally through *recursion operators* that depend on inverse differentiation. Integrable equations such as the sine-Gordon and cubic Schrödinger equations have an infinite number of such nonlocal symmetries.

1.2 Conservation Laws

In her celebrated 1918 paper [5], Emmy Noether showed that if a DE system admits a variational principle, then any local transformation group leaving invariant the action integral for its Lagrangian density, i.e., a *variational symmetry*, yields a *local conservation law*. Conversely, any local CL of a variational DE system arises from a variational symmetry, and hence there is a direct correspondence between local CLs and variational symmetries (Noether's theorem).

However there are limitations in the use of Noether's theorem.

- Its application is restricted to variational systems. In particular, a given DE system, *as written*, is variational if and only if its linearized system is self-adjoint.
- One has the difficulty of finding local symmetries of the action integral. In general, not all local symmetries of a variational DE system are variational symmetries.
- The use of Noether's theorem to find local conservation laws is coordinate-dependent.

The *Direct Method* for finding CLs allows one to find local CLs more generally for a given DE system. A CL of a given DE system is a divergence expression that vanishes on all solutions of the DE system. Local CLs arise from scalar products formed by linear combinations of *local CL multipliers* (factors that are functions of independent and dependent variables and their derivatives) multiplying each DE in the system. This scalar product is annihilated by the Euler operators associated with each of its dependent variables without restricting these variables in the scalar product to solutions of the system of DEs, i.e., the dependent variables are replaced by arbitrary functions of the independent variables.

If a given DE system, *as written*, is variational, then local CL multipliers correspond to variational symmetries. In the variational situation, using the Direct Method, local CL multipliers satisfy a linear system of determining equations that includes the linearizing system of the given DE system augmented by additional determining equations that taken together correspond to the action integral being invariant under the associated variational symmetry.

More generally, in using the Direct Method for any given DE system, the local CL multipliers are the solutions of an easily found linear determining system that includes the adjoint system of the linearizing DE system [1, 2, 9–11].

For any set of local CL multipliers, usually one can directly find the fluxes and density of the corresponding local CL and, if this proves difficult, there is an integral formula that yields them without the need of a specific functional (Lagrangian) even in the case when the given DE system is variational [9–11].

One can compare the number of local symmetries and the number of local CLs of a given DE system. When a DE system is variational, i.e., its linearized system

is self-adjoint, then local CLs arise from a subset of its local symmetries and the number of linearly independent local CLs cannot exceed the number of higher-order symmetries. In general, this will not be the case when a system is not variational. Here a given DE system can have more local conservation laws than local symmetries as well as vice versa.

For any given DE system, a transformation group (continuous or discrete) that leaves it invariant yields an explicit formula that maps a CL to a CL of the same system, whether or not the given system is variational. If the transformation group is a one-parameter Lie group of point (or contact) transformations, then in terms of a parameter expansion a given CL can map into more than one additional CL for the given DE system [2, 24].

1.3 Nonlocally Related Systems and Nonlocal Symmetries

Systematic procedures have been found to seek nonlocal symmetries of a given PDE system through applying Lie's algorithm to nonlocally related systems. In particular, to apply symmetry methods to PDE systems, one needs to work in some specific coordinate frame in order to perform calculations. A procedure to find symmetries that are nonlocal and yet are local in some related coordinate frame involves embedding a given PDE system in another PDE system obtained by adjoining nonlocal variables in such a way that the resulting nonlocally related PDE system is equivalent to the given system. Consequently, any local symmetry of the nonlocally related system yields a symmetry of the given system (The converse also holds). A local symmetry of the nonlocally related system, with the corresponding infinitesimals for the variables of the given PDE system having an essential dependence on nonlocal variables, yields a nonlocal symmetry of the given PDE system.

There are two known systematic ways to find such an embedding.

- Each local CL of a given PDE system yields a nonlocally related system. For each local CL, one can introduce a potential variable(s). Here the nonlocally related system is the given PDE system augmented by a corresponding potential system [2, 25–27].
- Each point symmetry of a given PDE system yields a nonlocally related system. Here, as a first step, the given PDE system naturally yields a locally related PDE system (intermediate system) arising from the canonical coordinates of the point symmetry. In turn, the intermediate system has a natural CL which yields a nonlocally related system (inverse potential system) for the given PDE system [28, 29]. The intermediate system plays the role of a potential system for the inverse potential system.

If a local symmetry of such a nonlocally related system has an essential dependence on nonlocal variables when projected to the given system, then it yields a nonlocal symmetry of the given PDE system. It turns out that many PDE systems have such systematically constructed nonlocal symmetries. Furthermore, one can

find additional nonlocal symmetries of a given PDE system through seeking local symmetries of an equivalent subsystem of the given system or one of its constructed nonlocally related systems provided that such a subsystem is nonlocally related to the given PDE system.

There are many applications of nonlocally related systems.

- Invariant solutions of nonlocally related systems (arising from CLs or point symmetries) can yield further solutions of a given PDE system.
- Since a point symmetry-based or CL-based nonlocal symmetry is a local symmetry of a constructed nonlocally related system, it generates a one-parameter family of solutions from any known solution (that is not an invariant solution) of such a nonlocally related system. In turn, this yields a one-parameter family of solutions from any known solution of the given PDE system.
- Local CLs of such nonlocally related systems can yield nonlocal CLs of a given PDE system if their local CL multipliers have an essential dependence on nonlocal variables.

Still wider classes of nonlocally related systems can be constructed systematically for a given PDE system. One can further extend embeddings through the effective use of local CLs to systematically construct trees of nonlocally related but equivalent PDE systems. If a given PDE system has n local CLs, then each CL yields potentials and corresponding potential systems. From the n local CLs, one can directly construct up to $2^n - 1$ independent nonlocally related systems of PDEs by considering corresponding potential systems individually (n singlets), in pairs ($n(n - 1)/2$ couplets), \dots , taken all together (one n -plet). Any of these systems could lead to the discovery of new nonlocal symmetries and/or nonlocal CLs of the given PDE system or any of the other nonlocally related systems. Such nonlocal CLs could yield further nonlocally related systems, etc. Furthermore, subsystems of such nonlocally related systems could yield further nonlocally related systems. Correspondingly, a tree of nonlocally related, and equivalent, systems is constructed for a given PDE system [2, 30, 31].

The situation in the case of multidimensional PDE systems (i.e., with at least three independent variables) is especially interesting. Here one can show that nonlocal symmetries and nonlocal CLs arising from the CL-based approach cannot arise from potential systems unless they are augmented by gauge constraints [2, 32, 33].

There exist many applications of such systematically constructed nonlocally related systems that further extend the use of symmetry methods for PDE systems.

- Through such constructions, one can systematically relate Eulerian and Lagrangian coordinate descriptions of gas dynamics and nonlinear elasticity. In particular, for the Eulerian coordinate description, a subsystem of the potential system arising from conservation of mass, naturally yields the corresponding description in Lagrangian coordinates [2, 30, 31, 34, 35].
- For a given class of PDEs with constitutive functions, one finds trees of nonlocally related systems yielding symmetries and CLs with respect to various forms of its constitutive functions.

- One can systematically seek noninvertible mappings of nonlinear PDE systems to linear PDE systems. Consequently, further nonlinear PDE systems can be mapped into equivalent linear PDE systems beyond those obtained through invertible mappings [2, 27, 36].
- One can systematically extend the class of linear PDE systems with variable coefficients that can be mapped into equivalent linear PDE systems with constant coefficients through inclusion of noninvertible mappings [2, 37, 38].

The rest of this chapter is organized as follows. In Sect. 2, we review local symmetries, Lie's algorithm to find local symmetries in evolutionary form, applications of local symmetries and as examples consider the heat equation and the Kortweg-de Vries equation. In Sect. 3, we consider the construction of conservation laws, introduce the Direct Method and its relationship to Noether's theorem, and show how symmetries could yield additional CLs from known CLs. As examples, we consider nonlinear telegraph equations, the Korteweg-de Vries equation, the Klein-Gordon equation, and nonlinear wave equations. In Sect. 4, we present systematic procedures to seek nonlocally related systems and nonlocal symmetries of a given PDE system with two independent variables. We introduce conservation law and point symmetry based methods as well as the use of subsystems to obtain trees of equivalent nonlocally related PDE systems. As examples, we focus on nonlinear wave equations, nonlinear telegraph equations, planar gas dynamics equations, and nonlinear reaction diffusion equations. In Sect. 5, we consider the situation of nonlocality in multidimensions. We show that if one directly applies the CL-based method to a single CL, then it is necessary to append a gauge constraint relating potential variables of the resulting vector potential system when seeking nonlocal symmetries. Some open problems are discussed.

2 Local Symmetries

Lie's algorithm for seeking point symmetries can be extended to seek more general local symmetries admitted by PDE systems. In the extension of Lie's algorithm, one uses differential consequences of the given PDE system, i.e., invariance of a given PDE system is understood to include its differential consequences. Here it is important to consider the infinitesimal generators for point symmetries in their *evolutionary form* where the independent variables are themselves invariant and the action of a group of point transformations is strictly an action on the dependent variables of the PDE system, so that solutions are *directly mapped into other solutions* under the group action. This allows one to readily extend Lie's algorithm to seek *contact symmetries* (only existing for scalar PDEs) where now the components of infinitesimal generators for dependent variables can depend at most on the first derivatives of the dependent variable of a given scalar PDE (if this dependence is at most linear on the first derivatives, then a contact symmetry is a point symmetry).