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V. Lakshmibai
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The Grassmannian Variety

Geometric and Representation-Theoretic Aspects

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Preface

This monograph represents an expanded version of a series of lectures given by V. Lakshmibai on Grassmannian varieties at the workshop on “Geometric Representation Theory” held at the Institut Teknologi Bandung in August 2011. While giving the lectures at the workshop, Lakshmibai realized the need for an introductory book on Grassmannian varieties that would serve as a good resource for learning about Grassmannian varieties, especially for graduate students as well as researchers who want to work in this area of algebraic geometry. Hence, the creation of this monograph.

This book provides an introduction to Grassmannian varieties and their Schubert subvarieties, focusing on the treatment of geometric and representation theoretic aspects. After a brief discussion on the basics of commutative algebra, algebraic geometry, cohomology theory, and Gröbner bases, the Grassmannian variety and its Schubert subvarieties are introduced. Following introductory material, the standard monomial theory for the Grassmannian variety and its Schubert subvarieties is presented. In particular, the following topics are discussed in detail:

- the construction of explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials in the Plücker coordinates (called standard monomials);
- the use of the standard monomial basis,
- the presentation of a proof of the vanishing of the higher cohomology groups of powers of the restriction of the tautological line bundle of the projective space (giving the Plücker embedding).

Further to using the standard monomial basis, the book discusses several geometric consequences, such as Cohen–Macaulayness, normality, unique factoriality, Gorenstein-ness, singular loci, etc., for Schubert varieties, and presents two kinds of degenerations of Schubert varieties, namely, degeneration to a toric variety, and degeneration to a monomial scheme. Additionally, the book presents the relationship between the Grassmannian and classical invariant theory. Included is a discussion on determinantal varieties—their relationship to Schubert varieties as well as to classical invariant theory. The book is concluded with a brief account of some topics related to the flag and Grassmannian varieties: standard monomial theory for a general G/Q, homology and cohomology of the flag variety, free resolutions of Schubert singularities, Bott–Samelson varieties, Frobenius splitting, affine flag varieties, and affine Grassmannian varieties.

This text can be used for an introductory course on Grassmannian varieties. The reader should have some familiarity with commutative algebra and algebraic geometry. A basic reference to commutative algebra is [21] and algebraic geometry [28]. The basic results from commutative algebra and algebraic geometry are summarized in Chapter 2. We have mostly used standard notation and terminology and have tried to keep notation to a minimum. Throughout the book, we have numbered theorems, lemmas, propositions, etc., in order according to their chapter and section; for example, 5.1.3 refers to the third item of the first section in the fifth chapter.

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Contents

Chapter 1
Introduction

This book is an expanded version of a series of lectures given by V. Lakshmibai on Grassmannian varieties at the workshop on "Geometric Representation Theory" held at the Institut Teknologi Bandung, Bandung, Indonesia, in August 2011. In this book, we have attempted to give a complete, comprehensive, and self-contained account of Grassmannian varieties and the Schubert varieties (inside a Grassmannian variety).

In algebraic geometry, Grassmannian varieties form an important fundamental class of projective varieties. In terms of importance, they are second only to projective spaces; in fact, a projective space itself is a certain Grassmannian. A Grassmannian variety sits as a closed subvariety of a certain projective space, the embedding being known as the celebrated Plücker embedding (as described in the next paragraph). Grassmannian varieties are important examples of homogeneous spaces; they are of the form $GL_n(K)/P$, P being a certain closed subgroup (for the Zariski topology) of $GL_n(K)$ (here, $GL_n(K)$ is the group of $n \times n$ invertible matrices with entries in the [base] field K which is supposed to be an algebraically closed field of arbitrary characteristic). In particular, a Grassmannian variety comes equipped with a $GL_n(K)$-action; in turn, the (homogeneous) coordinate ring (for the Plücker embedding) of a Grassmannian variety acquires a $GL_n(K)$-action, thus admitting representation-theoretic techniques for the study of Grassmannian varieties. Thus, Grassmannian varieties are at the crossroads of algebraic geometry, commutative algebra, and representation theory; their study is further enriched by their combinatorics. Schubert varieties in a Grassmannian variety form an important class of subvarieties, and provide a powerful inductive machinery for the study of Grassmannian varieties; in fact, a Grassmannian variety itself is a certain Schubert variety.

A Grassmannian variety (as a set) is the set of all subspaces of a given dimension d in K^n, for some $n \in \mathbb{N}$; it has a canonical projective embedding (the Plücker embedding) via the map which sends a d-dimensional subspace to the associated point in the projective space $\mathbb{P}(\Lambda^d K^n)$. A Grassmannian variety may be thought of

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as a partial flag variety: given a bunch of r distinct integers $\underline{d} := 1 \leq d_1 < d_2 < \ldots < d_r \leq n-1$, the *partial flag variety* $\mathcal{F}l_{\underline{d}}$, consists of partial flags of type $\underline{d}$, namely sequences $V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_r}$, V_{d_i} being a K-vector subspace of K^n of dimension d_i. The extreme case with $r = 1$, corresponds to the *Grassmannian variety* $G_{d,n}$ consisting of d-dimensional subspaces of K^n. If $d = 1$, then $G_{1,n}$ is just the $(n-1)$-dimensional projective space $\mathbb{P}_K^{n-1}$ (consisting of one-dimensional subspaces in K^n). For $r = n-1$, we get the celebrated *flag variety* $\mathcal{F}l_n$, consisting of flags in K^n, where a (full) flag is a sequence $(0) = V_0 \subset V_1 \subset \ldots \subset V_n = K^n$, $\dim V_i = i$. The flag variety $\mathcal{F}l_n$ has a natural identification with the homogeneous space $GL_n(K)/B$, B being the (Borel) subgroup of $GL_n(K)$ consisting of upper triangular matrices.

Throughout the 20^{th} century, mathematicians were interested in the study of the Grassmannian variety and its Schubert subvarieties (as well as the flag variety and its Schubert subvarieties). We shall now mention some of the highlights of the developments in the 20^{th} century on the Grassmannian and the flag varieties, pertaining to the subject matter of this book.

In 1934, *Ehresmann* (cf. [20]) showed that the classes of Schubert subvarieties in the Grassmannian give a $\mathbb{Z}$-basis for the cohomology ring of the Grassmannian, and thus established a key relationship between the geometry of the Grassmannian varieties and the theory of characteristic classes. This result of Ehresmann was generalized by *Chevalley* (cf. [14]) in 1956. Chevalley showed that the classes of the Schubert varieties (in the *generalized flag variety* G/B, G a semisimple algebraic group and B a Borel subgroup) form a $\mathbb{Z}$-basis for the Chow ring of the generalized flag variety. The results of Ehresmann and Chevalley were complemented by the work of *Hodge* (cf. [31, 32]). Hodge developed the *Standard Monomial Theory* for Schubert varieties in the Grassmannian. This theory consists in constructing explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials (called standard monomials) in the Plücker coordinates. Hodge's theory was generalized to G/B, for G classical by *Lakshmibai, Musili,* and *Seshadri* in the series Geometry of G/P I-V (cf. [49, 50, 53, 56, 82]) during 1975–1986; conjectures were then formulated (cf. [55]) by Lakshmibai and Seshadri in 1991 toward the generalization of Hodge's theory to exceptional groups. These conjectures were proved by *Littelmann* (cf. [65, 67, 68]) in 1994–1998, thus completing the standard monomial theory for semisimple algebraic groups. This theory has led to many interesting and important geometric and representation-theoretic consequences (see [44, 46, 48, 52, 54, 57, 65, 67, 68]).

In this book, we confine ourselves to the Grassmannian varieties and their Schubert subvarieties, since our goal is to introduce the readers to Grassmannian varieties fairly quickly, minimizing the technicalities along the way. We have attempted to give a complete and comprehensive introduction to the Grassmannian variety — its geometric and representation-theoretic aspects.

This book is divided into three parts. Part I is a brief discussion on the basics of commutative algebra, algebraic geometry, cohomology theory, and Gröbner bases. Part II is titled "Grassmann and Schubert Varieties." We introduce the Grassmannian variety and its Schubert subvarieties in Chapter 5. In Chapter 5, we also present the standard monomial theory for the Grassmannian variety and its Schubert subvarieties, namely construction of explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials in the Plücker coordinates (called standard monomials); we present a proof of the vanishing of the higher cohomology groups of powers of the restriction of the tautological line bundle of the projective space (giving the Plücker embedding), using the standard monomial basis. In Chapter 6, we deduce several geometric consequences—such as Cohen–Macaulayness, normality, a characterization for unique factoriality for Schubert varieties—in fact, these properties are established even for the cones over Schubert varieties (for the Plücker embedding). In addition, we describe the singular locus of a Schubert variety. In Chapter 7, we show that the generators for the ideal of the Grassmannian variety (as well as a Schubert variety) given by the Plücker quadratic relations give a Gröbner basis. We have also presented two kinds of degenerations of Schubert varieties, namely degeneration of a Schubert variety to a toric variety and degeneration to a monomial scheme. We also give a characterization for Gorenstein Schubert varieties.

Part III is titled "Flag Varieties and Related Varieties," and begins with Chapter 8, where we have included a brief introduction to flag varieties and statement of results on the standard monomial theory for flag varieties, as well as degenerations of flag varieties. In Chapter 9, we present the relationship between the Grassmannian and classical invariant theory. In Chapter 10, we present determinantal varieties, their relationship to Schubert varieties as well as to classical invariant theory. In Chapter 11, we give a brief account of some topics related to the flag and Grassmannian varieties: standard monomial theory for a general G/Q, homology and cohomology of the flag variety, free resolutions of Schubert singularities, Bott–Samelson varieties, Frobenius splitting, affine flag varieties, and affine Grassmannian varieties.

Part I
Algebraic Geometry: A Brief Recollection