

Developments in Mathematics

V. Lakshmibai
Justin Brown

The Grassmannian Variety

Geometric and Representation-Theoretic
Aspects

 Springer

Developments in Mathematics

VOLUME 42

Series Editors:

Krishnaswami Alladi, *University of Florida, Gainesville, FL, USA*

Hershel M. Farkas, *Hebrew University of Jerusalem, Jerusalem, Israel*

More information about this series at <http://www.springer.com/series/5834>

V. Lakshmibai • Justin Brown

The Grassmannian Variety

Geometric and Representation-Theoretic
Aspects

 Springer

V. Lakshmibai
Department of Mathematics
Northeastern University
Boston, MA, USA

Justin Brown
Department of Mathematics
Olivet Nazarene University
Bourbonnais, IL, USA

ISSN 1389-2177 ISSN 2197-795X (electronic)
Developments in Mathematics
ISBN 978-1-4939-3081-4 ISBN 978-1-4939-3082-1 (eBook)
DOI 10.1007/978-1-4939-3082-1

Library of Congress Control Number: 2015947780

Mathematics Subject Classification (2010): 14M15, 14L24, 13P10, 14D06, 14M12

Springer New York Heidelberg Dordrecht London
© Springer Science+Business Media New York 2015

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

Springer Science+Business Media LLC New York is part of Springer Science+Business Media (www.springer.com)

Preface

This monograph represents an expanded version of a series of lectures given by V. Lakshmibai on Grassmannian varieties at the workshop on “Geometric Representation Theory” held at the Institut Teknologi Bandung in August 2011. While giving the lectures at the workshop, Lakshmibai realized the need for an introductory book on Grassmannian varieties that would serve as a good resource for learning about Grassmannian varieties, especially for graduate students as well as researchers who want to work in this area of algebraic geometry. Hence, the creation of this monograph.

This book provides an introduction to Grassmannian varieties and their Schubert subvarieties, focusing on the treatment of geometric and representation theoretic aspects. After a brief discussion on the basics of commutative algebra, algebraic geometry, cohomology theory, and Gröbner bases, the Grassmannian variety and its Schubert subvarieties are introduced. Following introductory material, the standard monomial theory for the Grassmannian variety and its Schubert subvarieties is presented. In particular, the following topics are discussed in detail:

- the construction of explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials in the Plücker coordinates (called standard monomials);
- the use of the standard monomial basis,
- the presentation of a proof of the vanishing of the higher cohomology groups of powers of the restriction of the tautological line bundle of the projective space (giving the Plücker embedding).

Further to using the standard monomial basis, the book discusses several geometric consequences, such as Cohen–Macaulayness, normality, unique factoriality, Gorenstein-ness, singular loci, etc., for Schubert varieties, and presents two kinds of degenerations of Schubert varieties, namely, degeneration to a toric variety, and degeneration to a monomial scheme. Additionally, the book presents the relationship between the Grassmannian and classical invariant theory. Included is a discussion on determinantal varieties—their relationship to Schubert varieties as well as to classical invariant theory. The book is concluded with a brief account of some

topics related to the flag and Grassmannian varieties: standard monomial theory for a general G/Q , homology and cohomology of the flag variety, free resolutions of Schubert singularities, Bott–Samelson varieties, Frobenius splitting, affine flag varieties, and affine Grassmannian varieties.

This text can be used for an introductory course on Grassmannian varieties. The reader should have some familiarity with commutative algebra and algebraic geometry. A basic reference to commutative algebra is [21] and algebraic geometry [28]. The basic results from commutative algebra and algebraic geometry are summarized in Chapter 2. We have mostly used standard notation and terminology and have tried to keep notation to a minimum. Throughout the book, we have numbered theorems, lemmas, propositions, etc., in order according to their chapter and section; for example, 5.1.3 refers to the third item of the first section in the fifth chapter.

Acknowledgments: V. Lakshmibai thanks CIMPA, ICTP, UNESCO, MESR, MICINN (Indonesia Research School), as well as the organizers, Intan Muchtadi, Alexander Zimmermann of the workshop on “Geometric Representation Theory” held at the Institut Teknologi Bandung, Bandung, Indonesia, August 2011, for inviting her to give lectures on “Grassmannian Variety,” and also the Institute for the hospitality extended to her during her stay there.

Both authors thank Reuven Hodges for his feedback on some of the chapters. We thank the makers of ShareLaTeX for making this collaboration easier. We thank the referee for some useful comments, especially, pertaining to Chapter 11.

Finally, J. Brown thanks Owen, Evan, and Callen, for constantly reminding him just how fun life can be.

Boston, MA, USA
Bourbonnais, IL, USA

V. Lakshmibai
Justin Brown

Contents

1	Introduction	1
Part I Algebraic Geometry: A Brief Recollection		
2	Preliminary Material	7
2.1	Commutative Algebra	7
2.2	Affine Varieties	11
2.2.1	Zariski topology on \mathbb{A}^n	11
2.2.2	The affine algebra $K[X]$	13
2.2.3	Products of affine varieties	14
2.3	Projective Varieties	15
2.3.1	Zariski topology on \mathbb{P}^n	15
2.4	Schemes — Affine and Projective	16
2.4.1	Presheaves	16
2.4.2	Sheaves	16
2.4.3	Sheafification	17
2.4.4	Ringed and geometric spaces	17
2.5	The Scheme $\text{Spec}(A)$	18
2.6	The Scheme $\text{Proj}(S)$	19
2.6.1	The cone over X	20
2.7	Sheaves of \mathcal{O}_X -Modules	20
2.7.1	The twisting sheaf $\mathcal{O}_X(1)$	21
2.7.2	Locally free sheaves	22
2.7.3	The scheme $V(\Omega)$ associated to a rank n locally free sheaf Ω	22
2.7.4	Vector bundles	22
2.8	Attributes of Varieties	23
2.8.1	Dimension of a topological space	23
2.8.2	Geometric properties of varieties	24
2.8.3	The Zariski tangent space	24
2.8.4	The differential $(d\phi)_x$	25

3	Cohomology Theory	27
3.1	Introduction to Category Theory	27
3.2	Abelian Categories	28
3.2.1	Derived Functors	31
3.3	Enough Injective Lemmas	32
3.4	Sheaf and Local Cohomology	37
4	Gröbner Bases	39
4.1	Monomial Orders	39
4.2	Gröbner Basis	42
4.3	Compatible Weight Orders	43
4.4	Flat Families	46
Part II Grassmann and Schubert Varieties		
5	The Grassmannian and Its Schubert Varieties	51
5.1	Grassmannian and Flag Varieties	51
5.2	Projective Variety Structure on $G_{d,n}$	53
5.2.1	Plücker coordinates	53
5.2.2	Plücker Relations	54
5.2.3	Plücker coordinates as T -weight vectors	56
5.3	Schubert Varieties	57
5.3.1	Dimension of X_w	59
5.3.2	Integral Schemes	60
5.4	Standard Monomials	61
5.4.1	Generation by Standard Monomials	62
5.4.2	Linear Independence of Standard Monomials	63
5.5	Unions of Schubert Varieties	65
5.5.1	The Picard Group	67
5.6	Vanishing Theorems	67
6	Further Geometric Properties of Schubert Varieties	73
6.1	Cohen–Macaulay	73
6.2	Lemmas on Normality and Factoriality	77
6.2.1	Factoriality	82
6.3	Normality	85
6.3.1	Stability for multiplication by certain parabolic subgroups	86
6.4	Factoriality	88
6.5	Singular Locus	90
7	Flat Degenerations	95
7.1	Gröbner basis	95
7.2	Toric Degenerations	97
7.3	Monomial Scheme Degenerations	103
7.4	Application to the Degree of X_w	104
7.5	Gorenstein Schubert Varieties	108

Part III Flag Varieties and Related Varieties

8 The Flag Variety: Geometric and Representation Theoretic Aspects 117

8.1 Definitions 117

8.2 Standard Monomials on the Flag Variety 118

8.3 Toric Degeneration for the Flag Variety 121

8.4 Representation Theoretic Aspects 122

8.4.1 Application to $G_{d,n}$ 124

8.5 Geometric Aspects 124

8.5.1 Description of the tangent space 125

8.5.2 Pattern avoidance 125

9 Relationship to Classical Invariant Theory 129

9.1 Basic Definitions in Geometric Invariant Theory 129

9.1.1 Reductive Groups 130

9.2 Categorical Quotient 131

9.3 Connection to the Grassmannian 137

10 Determinantal Varieties 143

10.1 Determinantal Varieties 143

10.1.1 The determinantal variety D_t 143

10.1.2 Relationship between determinantal varieties and Schubert varieties 144

10.2 Standard Monomial Basis for $K[D_t]$ 145

10.2.1 The partial order \succeq 146

10.2.2 Cogeneration of an Ideal 147

10.3 Gröbner Bases for Determinantal Varieties 147

10.4 Connections with Classical Invariant Theory 149

10.4.1 The First and Second Fundamental Theorems of Classical Invariant Theory (cf. [88]) for the action of $GL_n(K)$ 150

11 Related Topics 155

11.1 Standard Monomial Theory for a General G/Q 155

11.2 The Cohomology and Homology of the Flag Variety 156

11.2.1 A \mathbb{Z} -basis for $H^*(Fl(n))$ 156

11.2.2 A presentation for the \mathbb{Z} -algebra $H^*(Fl(n))$ 156

11.2.3 The homology $H_*(Fl(n))$ 157

11.2.4 Schubert classes and Littlewood-Richardson coefficients 157

11.3 Free Resolutions 158

11.4 Bott–Samelson Scheme of G 158

11.5 Frobenius-Splitting 159

11.6 Affine Schubert Varieties 160

11.7 Affine Flag and Affine Grassmannian Varieties 161

References 163

List of Symbols 167

Index 169

Chapter 1

Introduction

This book is an expanded version of a series of lectures given by V. Lakshmibai on Grassmannian varieties at the workshop on “Geometric Representation Theory” held at the Institut Teknologi Bandung, Bandung, Indonesia, in August 2011. In this book, we have attempted to give a complete, comprehensive, and self-contained account of Grassmannian varieties and the Schubert varieties (inside a Grassmannian variety).

In algebraic geometry, Grassmannian varieties form an important fundamental class of projective varieties. In terms of importance, they are second only to projective spaces; in fact, a projective space itself is a certain Grassmannian. A Grassmannian variety sits as a closed subvariety of a certain projective space, the embedding being known as the celebrated Plücker embedding (as described in the next paragraph). Grassmannian varieties are important examples of homogeneous spaces; they are of the form $GL_n(K)/P$, P being a certain closed subgroup (for the Zariski topology) of $GL_n(K)$ (here, $GL_n(K)$ is the group of $n \times n$ invertible matrices with entries in the [base] field K which is supposed to be an algebraically closed field of arbitrary characteristic). In particular, a Grassmannian variety comes equipped with a $GL_n(K)$ -action; in turn, the (homogeneous) coordinate ring (for the Plücker embedding) of a Grassmannian variety acquires a $GL_n(K)$ -action, thus admitting representation-theoretic techniques for the study of Grassmannian varieties. Thus, Grassmannian varieties are at the crossroads of algebraic geometry, commutative algebra, and representation theory; their study is further enriched by their combinatorics. Schubert varieties in a Grassmannian variety form an important class of subvarieties, and provide a powerful inductive machinery for the study of Grassmannian varieties; in fact, a Grassmannian variety itself is a certain Schubert variety.

A Grassmannian variety (as a set) is the set of all subspaces of a given dimension d in K^n , for some $n \in \mathbb{N}$; it has a canonical projective embedding (the Plücker embedding) via the map which sends a d -dimensional subspace to the associated point in the projective space $\mathbb{P}(\Lambda^d K^n)$. A Grassmannian variety may be thought of

as a partial flag variety: given a bunch of r distinct integers $\underline{d} := 1 \leq d_1 < d_2 < \dots < d_r \leq n - 1$, the *partial flag variety* $\mathcal{F}l_{\underline{d}}$, consists of partial flags of type \underline{d} , namely sequences $V_{d_1} \subset V_{d_2} \subset \dots \subset V_{d_r}$, V_{d_i} being a K -vector subspace of K^n of dimension d_i . The extreme case with $r = 1$, corresponds to the *Grassmannian variety* $G_{d,n}$ consisting of d -dimensional subspaces of K^n . If $d = 1$, then $G_{1,n}$ is just the $(n - 1)$ -dimensional projective space \mathbb{P}_K^{n-1} (consisting of one-dimensional subspaces in K^n). For $r = n - 1$, we get the celebrated *flag variety* $\mathcal{F}l_n$, consisting of flags in K^n , where a (full) flag is a sequence $(0) = V_0 \subset V_1 \subset \dots \subset V_n = K^n$, $\dim V_i = i$. The flag variety $\mathcal{F}l_n$ has a natural identification with the homogeneous space $GL_n(K)/B$, B being the (Borel) subgroup of $GL_n(K)$ consisting of upper triangular matrices.

Throughout the 20th century, mathematicians were interested in the study of the Grassmannian variety and its Schubert subvarieties (as well as the flag variety and its Schubert subvarieties). We shall now mention some of the highlights of the developments in the 20th century on the Grassmannian and the flag varieties, pertaining to the subject matter of this book.

In 1934, *Ehresmann* (cf. [20]) showed that the classes of Schubert subvarieties in the Grassmannian give a \mathbb{Z} -basis for the cohomology ring of the Grassmannian, and thus established a key relationship between the geometry of the Grassmannian varieties and the theory of characteristic classes. This result of Ehresmann was generalized by *Chevalley* (cf. [14]) in 1956. Chevalley showed that the classes of the Schubert varieties (in the *generalized flag variety* G/B , G a semisimple algebraic group and B a Borel subgroup) form a \mathbb{Z} -basis for the Chow ring of the generalized flag variety. The results of Ehresmann and Chevalley were complemented by the work of *Hodge* (cf. [31, 32]). Hodge developed the *Standard Monomial Theory* for Schubert varieties in the Grassmannian. This theory consists in constructing explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials (called standard monomials) in the Plücker coordinates. Hodge's theory was generalized to G/B , for G classical by *Lakshmibai*, *Musili*, and *Seshadri* in the series *Geometry of G/P I-V* (cf. [49, 50, 53, 56, 82]) during 1975–1986; conjectures were then formulated (cf. [55]) by Lakshmibai and Seshadri in 1991 toward the generalization of Hodge's theory to exceptional groups. These conjectures were proved by *Littelmann* (cf. [65, 67, 68]) in 1994–1998, thus completing the standard monomial theory for semisimple algebraic groups. This theory has led to many interesting and important geometric and representation-theoretic consequences (see [44, 46, 48, 52, 54, 57, 65, 67, 68]).

In this book, we confine ourselves to the Grassmannian varieties and their Schubert subvarieties, since our goal is to introduce the readers to Grassmannian varieties fairly quickly, minimizing the technicalities along the way. We have attempted to give a complete and comprehensive introduction to the Grassmannian variety — its geometric and representation-theoretic aspects.

This book is divided into three parts. Part I is a brief discussion on the basics of commutative algebra, algebraic geometry, cohomology theory, and Gröbner bases. Part II is titled “Grassmann and Schubert Varieties.” We introduce the Grassmannian

variety and its Schubert subvarieties in Chapter 5. In Chapter 5, we also present the standard monomial theory for the Grassmannian variety and its Schubert subvarieties, namely construction of explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials in the Plücker coordinates (called standard monomials); we present a proof of the vanishing of the higher cohomology groups of powers of the restriction of the tautological line bundle of the projective space (giving the Plücker embedding), using the standard monomial basis. In Chapter 6, we deduce several geometric consequences—such as Cohen–Macaulayness, normality, a characterization for unique factoriality for Schubert varieties—in fact, these properties are established even for the cones over Schubert varieties (for the Plücker embedding). In addition, we describe the singular locus of a Schubert variety. In Chapter 7, we show that the generators for the ideal of the Grassmannian variety (as well as a Schubert variety) given by the Plücker quadratic relations give a Gröbner basis. We have also presented two kinds of degenerations of Schubert varieties, namely degeneration of a Schubert variety to a toric variety and degeneration to a monomial scheme. We also give a characterization for Gorenstein Schubert varieties.

Part III is titled “Flag Varieties and Related Varieties,” and begins with Chapter 8, where we have included a brief introduction to flag varieties and statement of results on the standard monomial theory for flag varieties, as well as degenerations of flag varieties. In Chapter 9, we present the relationship between the Grassmannian and classical invariant theory. In Chapter 10, we present determinantal varieties, their relationship to Schubert varieties as well as to classical invariant theory. In Chapter 11, we give a brief account of some topics related to the flag and Grassmannian varieties: standard monomial theory for a general G/Q , homology and cohomology of the flag variety, free resolutions of Schubert singularities, Bott–Samelson varieties, Frobenius splitting, affine flag varieties, and affine Grassmannian varieties.

Part I
Algebraic Geometry: A Brief Recollection