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Gijs M. Tuynman

Supermanifolds and Supergroups

Basic Theory

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Preface

This book is a self contained introduction to super differential geometry, intended for graduate students in mathematics and theoretical physics and other people who want to learn the basics about supermanifolds. It is self contained in that it only requires standard undergraduate knowledge. However, some knowledge of ordinary (non super) differential geometry will make this text much easier to read.

Various versions of super differential geometry exist, some of which are equivalent and some of which are not. The version presented here is equivalent to those that are most widely used: the H^∞ supermanifolds of DeWitt and the sheaf theoretic approach to supermanifolds of Kostant and Leites. The approach taken here is based on an index free formalism using a graded commutative ring $\mathcal{A}$ containing the usual real numbers as well as so called anticommuting numbers. Starting with a non-standard definition of a differentiable function, valid in the real case, in the complex case and in the super case, the theory is developed as if it were ordinary differential geometry. It is shown that most constructions and theorems in ordinary differential geometry have a natural generalization to the super context. Moreover, even the proofs bear more than a superficial resemblance to their counter parts in ordinary differential geometry. The (equivalent) sheaf-theoretic approach to supermanifolds makes it manifest that the theory is “independent” of the choice of $\mathcal{A}$, but at the same time it hides the more geometric nature of the theory. The approach presented here can be seen as a theory with a parameter $\mathcal{A}$. Choosing $\mathcal{A} = \mathbf{R}$ gives ordinary differential geometry, choosing $\mathcal{A} = \bigwedge \mathbf{R}^\infty$ gives super differential geometry, choosing $\mathcal{A} = \mathbf{C}$ gives the theory of complex manifolds, etc. Of course, in each of these cases some small but usually superficial changes have to be made, and not all results remain true in all cases (e.g., Batchelor’s theorem, which uses partitions of unity, is not valid for (super) complex manifolds). But the main body of the results is not affected by the choice of $\mathcal{A}$.

In Chapter I the general theory of graded linear algebra (graded by an arbitrary abelian group) is outlined. This plays the same role in super differential geometry as does linear algebra in ordinary differential geometry and as does commutative algebra in algebraic geometry. Since the basic ring is (in principle) not commutative, we have to make a distinction between left and right linear maps. The isomorphism between these two kinds of maps is given by the operator $\mathfrak{T}$, which will later be identified with (super) transposition of matrices.

In chapter II we specialize to $\mathbf{Z}_2$-graded linear algebra and we impose some restrictions on the ring $\mathcal{A}$. The canonical example of $\mathcal{A}$ that satisfies all conditions (the ones imposed in chapter II and also other ones imposed later on) is the exterior algebra of an infinite dimensional real vector space: $\mathcal{A} = \bigwedge \mathbf{R}^\infty$. Some of the more important points of this chapter are the following. In section 2 it is shown that any (finitely generated) free graded $\mathcal{A}$-module admits a well defined graded dimension. In section 4 the relation between matrices and linear maps is explained. The reader should really pay attention here, because there are *three* different natural ways to associate a matrix to a linear map, and these three different ways imply different ways how to multiply a matrix by an element of $\mathcal{A}$ (so as to be compatible with the multiplication of the corresponding linear map by the element of $\mathcal{A}$). It is here that we see most clearly the role of the transposition operator introduced in chapter I to relate left and right linear maps. In section 5 the graded trace is defined for any linear map (and thus for any matrix, not only the even ones), as well as its integrated version for even maps, the graded determinant or Berezinian. Finally in section 6 the body map $\mathbf{B}$ is introduced, which provides an "isomorphism" between equivalence classes of free graded $\mathcal{A}$-modules and direct sums of two real vector spaces. It is this body map which gives the link between standard linear algebra and $\mathbf{Z}_2$-graded linear algebra.

The heart of this book lies in chapter III, in which the notion of a supermanifold is developed based on a non-standard definition of differentiable functions. The key idea is expressed by the following formula, valid for functions f of class C^1 on convex domains in $\mathbf{R}^n$:

$$f(x) - f(y) = \left(\int_0^1 f'(sx + (1-s)y)\, ds \right) \cdot (x - y) \, .$$

If we write this as $f(x) - f(y) = g(x,y) \cdot (x - y)$, it is obvious that f is of class C^1 if and only if the function g is of class C^0. Moreover, if a g with this property exists, it is also easy to see that f', the derivative of f, is given by $f'(x) = g(x,x)$. If we now note that the formula $f(x) - f(y) = g(x,y) \cdot (x - y)$ does not involve quotients nor limits, we can apply the same definition to super functions, for which there generally do not exist quotients (because of nilpotent elements in $\mathcal{A}$), nor does the natural topology (the DeWitt topology) admit unique limits (being non Hausdorff). Based on this idea, smooth functions (C^∞) on super domains with p even coordinates and q odd coordinates are defined. It is shown, using the body map $\mathbf{B}$ defined in chapter II, that these smooth functions are in bijection with ordinary smooth real-valued functions of p real variables, multiplied by antisymmetric polynomials in q variables. This result is usually taken as the definition of smooth super functions; here it is a consequence of a more general definition, a definition which applies as well to ordinary functions as to super functions. The last two sections of chapter III are devoted to copying the standard definition of manifolds in terms of charts and transition functions to the case in which the transition functions are super smooth functions.

In chapter IV the general theory of fiber and vector bundles is developed. The first two sections deal with general fiber bundles and how to construct new ones out of given ones. The next two sections deal with vector bundles and how to generalize the construction of new $\mathcal{A}$-modules to the setting of vector bundles. In section 5 the behavior of the operation of taking sections under the various operations one can perform on $\mathcal{A}$-modules is considered. In section 6 the exterior algebra of a (dual) bundle is discussed in more detail, as well as the pull-back of sections. The main purpose of sections 5 and 6 is to provide a rigorous justification for operations everybody performs without thinking twice. In section 7 one finds a proof of part of the Serre-Swann theorem that the module of sections of a vector bundle is a finitely generated projective module over the ring of smooth functions on the base manifold. The proof of this result needs the notion of a metric on a free graded $\mathcal{A}$-module, a notion whose definition is subtly different from what one would expect. These results are not used elsewhere, but they are needed to complete the proofs of statements given in section 5. The last section in chapter IV on Batchelor's theorem merits ample attention. This theorem says that any supermanifold is "isomorphic" to an ordinary vector bundle over an ordinary manifold, or, stated differently, for any smooth supermanifold there exists an atlas in which the transition functions are of the special form: even coordinates depend on even coordinates only, and odd coordinates depend in a linear way on odd coordinates. The proof is "constructive" in that it provides an explicit algorithm to compute such an atlas given an arbitrary atlas. The quotes are needed because this algorithm requires a partition of unity on the underlying ordinary manifold.

Chapter V treats the standard machinery of differential geometry. In section 1 the tangent bundle is defined and it is shown that sections of it, called vector fields, are equivalent to derivations of the ring of smooth functions. In section 2 the tangent map is defined, which in turn gives rise to the notions of immersion and embedding. In section 3 the relationship between the tangent map and the derivative of a map are studied in more detail. It turns out that in the super case this is in general not a 1–1 correspondence. Generalizing the notion of the derivative of an $\mathcal{A}$-valued function to vector bundle valued functions, a necessary and sufficient condition is given for a vector bundle to be trivial as a vector bundle. Here one also can find an example of a vector bundle which is trivial as fiber bundle, but not as vector bundle. Sections 4 and 5 then concentrate on the notion of the flow of a vector field and the well known proposition that two vector fields commute if and only if their flows commute. For odd vector fields this amounts to saying that an odd vector field is integrable if and only if its auto commutator is zero. Section 6 treats Frobenius' theorem on integrability of subbundles of the tangent bundle, the notion of integral manifolds and the existence of leaves for a foliation. In section 7 the calculus of (exteriordifferential) k-forms is given, including the definition of the Lie derivative and its relation with the flow of a vector field. Finally in section 8 an elementary proof is given of the fact that the de Rham cohomology of a supermanifold is the same as that of the underlying ordinary manifold (its body).

Chapter VI treats the basic facts about super Lie groups and their associated super Lie algebras. In section 1 one finds the basic definition of a super Lie group and the construction of the associated super Lie algebra. The exponential map from the super Lie algebra to the super Lie group is defined in section 2. There one also finds the proof that it intertwines a homomorphism of super Lie groups and its induced morphism on the associated super Lie algebras. Section 3 is rather technical and computes the derivative of the exponential map. Section 4 deals with the relationship between Lie subgroups and Lie subalgebras, whereas section 5 treats homogeneous supermanifolds. Section 6 is again technical and proves that any smooth action can be transformed into a pseudo effective action. The last section gives a geometric proof that to each finite dimensional super Lie algebra corresponds a super Lie group.

Chapter VII is more advanced and discusses the general concept of a connection on a fiber bundle. Sections 1 and 8 are technical and provide the necessary theory of vector valued and vector bundle valued differential forms. In section 2 the general concept of an Ehresmann connection is introduced, as well as the more restrictive notion of FVF connection, which is an Ehresmann connection determined by the fundamental vector fields of the structure group on the typical fiber. FVF connections have nice properties: they are defined on any fiber bundle, they include the standard examples of connections such as the (principal) connection on a principal fiber bundle and linear connections on vector bundles, and they always allow parallel transport. In sections 3 and 4 the particular case of an FVF connection on a principal fiber bundle is studied, which includes the description by a connection 1-form, the exterior covariant derivative and a discussion about the curvature 2-form. In section 5 it is shown that any FVF connection can be seen as induced by an FVF connection on a principal fiber bundle. Sections 6 and 7 treat the notion of a covariant derivative on a vector bundle and prove that it is equivalent to an FVF connection. It includes the proof that the covariant derivative measures how far away a (local) section is from being horizontal. In sections 9 and 10 the covariant derivative on a vector bundle is generalized to vector bundle valued differential forms and it is shown how the exterior covariant derivative (on a principal fiber bundle), the ordinary exterior derivative of differential forms and the generalized covariant derivative (on a vector bundle) are intimately related.

This book is written in a logical order, meaning that a proof of a statement never uses future results and meaning that related subjects are put together. This is certainly not the most pedagogical way to present the subject, but it avoids the risk of circular arguments. As a consequence, the novice reader should not read this book in a linear order. For a first reading, one can easily skip sections 7 and 8 of chapter I. From chapter IV one should certainly read sections 1–3, but coming back for sections 4–6 (and then only superficially) just before starting to read section 6 of chapter V. The reader who already has a working knowledge of ordinary manifold theory need not read all sections with the same attention and at a first reading (s)he can even skip chapter IV completely.

One final word on terminology: in this introduction I have systematically used the adjective *super*. On the other hand, in the main text I never use this adjective, but rather the prefix $\mathcal{A}$. The reason to do so is that one should regard this theory not as opposed to ordinary differential geometry (super versus non-super), but more as a theory with a parameter $\mathcal{A}$, indicating over which ring it is developed.

In preparing chapters I–VI I have relied heavily on the first three chapters of F. Warner's classic "Foundations of Differentiable Manifolds and Lie Groups,", while chapter VII is based on H. Pijls' review article "The Yang-Mills equations." Other sources of inspiration have been the first volume of M. Spivak's "A Comprehensive Introduction to Differential Geometry" and "Les Tenseurs" of L. Schwartz. During the years it took me to write this book, I have benefitted from the hospitality of the following three institutions: MSRI (Berkeley, USA), CPT (Marseille, France) and LNCC (Rio de Janeiro, Brazil). Special thanks are due to P. Bongaarts for some excellent suggestions concerning chapter I and to V. Thilliez who helped me with [III. 1.12]. Finally, I am convinced I got the idea for [IV.7.3] from a paper by S. Sternberg, but I can no longer find the source.

Lille, january 2004

Chapter I

$\mathfrak{A}$-graded commutative linear algebra

Linear algebra is concerned with the study of vector spaces over the real numbers (or more generally over a field) and linear maps. A standard course on linear algebra more or less starts with the introduction of the concept of a basis. Immediately afterwards one usually restricts attention to finite dimensional vector spaces. Next on the list is the concept of a subspace and with that notion one derives some elementary properties of linear maps. Then one introduces bilinear maps, with a scalar product as the most important example. This gives rise to the notions of orthogonal basis, orthogonal linear map, and orthogonal subspaces, eventually followed by a classification of quadrics. More advanced courses treat the notions of multilinear maps, tensor products, and exterior powers. Algebras, and in particular Lie algebras, are usually treated separately.

Besides analysis, these concepts in linear algebra form the basis of differential geometry. One could even say that differential geometry is the interplay between analysis and linear algebra. Algebraic geometry is closely related to differential geometry, but hardly relies on analysis; it is mainly concerned with algebraic structures. For that it needs a generalization of linear algebra in which a vector space over a field is replaced by a module over a commutative ring with unit. Commutative algebra is the theory which plays in algebraic geometry the same role as linear algebra does in differential geometry. In commutative algebra the notion of basis more or less disappears, but subspaces, tensor products, and exterior powers can still be defined.

In supergeometry one replaces the field of real numbers, not by a commutative ring, but by a graded commutative ring. Since such a ring is not commutative, commutative algebra does not apply. In this context, graded means $\mathbf{Z}_2$-graded, i.e., the ring and all modules are a direct sum of two subspaces, the even and odd parts. In this first chapter we look at an even more general situation. We denote by $\mathfrak{A}$ an arbitrary abelian group and we denote by $\mathcal{A}$ an arbitrary $\mathfrak{A}$-graded commutative ring with unit $1 \neq 0$ (i.e., a ring which splits as a direct sum of subspaces indexed by $\mathfrak{A}$ and satisfying conditions how these subspaces commute). We will show that all concepts of linear algebra that are important *for differential geometry can be generalized to $\mathfrak{A}$-graded commutative linear algebra, i.e., to the theory of $\mathfrak{A}$-graded $\mathcal{A}$-modules.*

1. $\mathfrak{A}$-GRADED COMMUTATIVE RINGS AND $\mathfrak{A}$-GRADED $\mathcal{A}$-MODULES

In this first section we give the definitions of the principal objects of this book: $\mathfrak{A}$-graded commutative algebras and $\mathfrak{A}$-graded $\mathcal{A}$-modules. $\mathfrak{A}$-graded $\mathcal{A}$-modules are a special kind of $\mathfrak{A}$-graded $\mathcal{A}$-bimodules, a fact that will greatly facilitate constructions of new $\mathfrak{A}$-graded $\mathcal{A}$-modules, one of which is discussed in this section: the $\mathfrak{A}$-graded submodule.

1.1 Definition. Given abelian groups $G_1, \ldots, G_k$ and H, a map $\phi : G_1 \times \cdots \times G_k \to H$ is called *k-additive* if for all i and for all $g_i, \widehat{g}_i \in G_i$ we have:

$$\phi(g_1, \ldots, g_{i-1}, g_i + \widehat{g}_i, g_{i+1}, \ldots, g_k) = \phi(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_k) + \phi(g_1, \ldots, g_{i-1}, \widehat{g}_i, g_{i+1}, \ldots, g_k) \ .$$

1.2 Definition. Let G be an abelian group and let $G_i, i \in I$ be a family of subgroups. One writes $G = \bigoplus_{i\in I} G_i$ if and only if for each element $g \in G$ there exist *unique* $g_i \in G_i$, only finitely many of them non-zero, such that $g = \sum_{i\in I} g_i$; it is called *the (unique) decomposition of g into G_i-components.*

1.3 Definitions. • Let $\mathcal{A}$ be a ring. A *left module over the ring* $\mathcal{A}$ (or a *left $\mathcal{A}$-module*) is an abelian group E equipped with a map $m_L : \mathcal{A} \times E \to E$ that is bi-additive and satisfies

$$m_L(a, m_L(b, e)) = m_L(ab, e) \ .$$

This map is called *left multiplication by elements of $\mathcal{A}$*, and (as is usual) we will omit the symbol m_L if no confusion is possible and just write ae or $a \cdot e$ for $m_L(a, e)$. If $\mathcal{A}$ contains a unit $1 \neq 0$, we also require that $1 \cdot e = e$ for all $e \in E$. In a similar way, a *right $\mathcal{A}$-module* is an abelian group E equipped with a map (right multiplication) $m_R : E \times \mathcal{A} \to E$ that is bi-additive and satisfies $m_R(m_R(e, a), b) = m_R(e, ab)$. And as before, if no confusion is possible we will just write ea or $e \cdot a$ for $m_R(e, a)$. As for left $\mathcal{A}$-modules, if $\mathcal{A}$ contains a unit $1 \neq 0$, we require that $e \cdot 1 = e$ for all $e \in E$. Since $\mathcal{A}$ is in general not commutative, the notions of left and right $\mathcal{A}$-modules do not coincide.

• An *$\mathcal{A}$-bimodule* is an abelian group E which is at the same time a left and a right $\mathcal{A}$-module such that the left and right actions commute, i.e., for all $a, b \in \mathcal{A}$, $e \in E$: $m_L(a, m_R(e, b)) = m_R(m_L(a, e), b)$, which can also be written as $(ae)b = a(eb)$.

• A subset F of a left/right $\mathcal{A}$-module E is called a *submodule* if F is a subgroup with respect to the additive structure of E such that $\mathcal{A}F \subset F$. It follows that F, with the induced multiplication of $\mathcal{A}$, is itself a left/right $\mathcal{A}$-module.