

Elena Nardi

# Amongst Mathematicians

Teaching and Learning Mathematics  
at the University Level

# **Amongst Mathematicians**

# MATHEMATICS TEACHER EDUCATION

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## VOLUME 3

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Elena Nardi  
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# Amongst Mathematicians

Teaching and Learning Mathematics at  
University Level

 Springer

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*'I was formed by nature to be a mathematically curious entity:  
not one but half of two'*  
[the character of] Niels Bohr in *Copenhagen* by Michael Frayn

*'La rêve de l'individu est d' être deux; la rêve de l'état est d' être seul.'*  
[The dream of the individual is to be two; the dream of the state is to be on its own.]  
Jean-Luc Godard, *Notre Musique*

## ACKNOWLEDGEMENTS

The main bulk of this book consists of fictional, yet data-grounded, dialogues between two characters: M, mathematician, and RME, researcher in mathematics education. The discussion in these dialogues is triggered by samples of the writing or speaking of mathematics undergraduates at the beginning of their university studies. The data on which these dialogues are grounded, from mathematicians and their students, was collected over a long period: from 1992, when I embarked on my doctoral studies at Oxford, to 2004, when the last study reported in this book was completed. To all of these mathematicians and students I extend my warmest thanks – particularly those mathematicians from the latest study who ‘became’ M more directly: thank you for giving your valuable time and energy and for trusting me with your words. I hope you will find that, however playful, the ‘processing’ of your words was never done with anything less than respect for the complexity of the ideas you intended to put across and a desire for understanding.

The ‘playfulness’ I cite above refers to the somewhat unconventional choice of format for mediating the ideas in the book, the dialogue. Given my origins as a mathematician, this choice may appear surprising. For a mathematician however who has always been fond of words and ...wordiness of all sorts, I am lucky to have found myself in an environment where this penchant – for telling stories that aspire to reconcile vividness with subtlety and rigour – was treated as a methodological ...oddy that was not only never discouraged; it was allowed to grow: University of East Anglia’s School of Education has a long tradition of nurturing this kind of openness and I have been counting myself a fortunate beneficiary of this tradition since 1998.

Since 2000 I have been working closely with Paola Iannone whose commitment, talent and generosity have helped shape the most essential of the studies on which I draw in this book. I am profoundly grateful.

Finally a word for my friends and family\*. For years now, with energy and method, we have managed a far from trivial feat: keeping the tie alive and crucial across several cities and countries, and two continents. For proving that distance can be reduced to nothing more than a construct of geography this book is for you. If there is any wisdom in the statement that a step into the unknown is taken by people either who have nothing to lose or who have solid safety nets to fall back into when the step proves too tough or foolhardy, then your faith, unflinching and energising, makes my case fit the latter. Thank you. Always.

---

\* I felt that a listing of names at this stage would dull the intended effect: expressing my deeply felt gratitude. The story behind this work is told in several places across the book, mostly in Chapter 2 and in the Post-script. Most of the names that should be listed here appear, in footnotes, within the ‘scenes’ of the story there – namely at the *heart* of the story, where they rightfully belong.

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## PROLOGUE

The main part of this book, **Chapters 3-8**, consists of dialogues between two characters M, a mathematician, and RME, a researcher in mathematics education. These dialogues focus on a range of issues regarding the learning and teaching of mathematics at university level. Each dialogue starts from a discussion of a sample of data (students' writing or speaking) that exemplify these issues. Both the samples that M and RME discuss as well as their discussions are grounded in data collected in the course of several studies that I have been involved in since 1992. In **Chapter 1** I outline the studies that formed the raw material for the book as well as the wider area of mathematics education research these studies are embedded in and the book aims to contribute to. In **Chapter 2** I outline the processing that the data collected in these previous studies has gone through in order to reach the dialogic form in which it is presented in Chapters 3-8. **Chapter 3** focuses on students' mathematical reasoning and in particular their conceptualization of the necessity for Proof and their enactment of various proving techniques. **Chapter 4** shifts the focus towards the students' mathematical expression and their attempts to mediate mathematical meaning through words, symbols and diagrams. **Chapters 5 and 6** offer accounts of the students' encounter with some fundamental concepts of advanced mathematics – Functions (across the domains of Analysis, Linear Algebra and Group Theory) and Limits. **Chapter 7** revisits many of the 'learning stories' told in Chapters 3-6 in order to highlight issues of university-level pedagogy. Finally, in **Chapter 8** M and RME, starting from the experience of working together in the context of the studies on which the book is based – and as showcased in Chapters 3-7 – discuss their often fragile relationship as well as the necessary and sufficient conditions for a collaboration between mathematicians and researchers in mathematics education. The book concludes with: a brief **Epilogue** in which I reflect on the experience of engaging with the research behind this book, and with its production, and I outline some steps that, at the final stages of writing, this research was beginning to take; and, with a **Post-script** in which I offer a chronological and reflexive account of the events that led to the production of the book.

**Note to the reader regarding Chapters 1 and 2.** Chapter 1 describes the theoretical background and previous studies on which the book is based and Chapter 2 the method through which the dialogues in Chapters 3-8 came to be. Readers less interested in these may wish to skip these chapters and go directly to Chapter 3. For minimal familiarisation there is a one-page summary at their beginning.

**Note to the reader regarding Chapters 3-8.** Each of these consists of Episodes which I recommend that you read as follows: engage briefly with the mathematics in the Episode (problem, solution and examples of student response); reflect on the learning/teaching issues these may generate; and, read and reflect on the dialogue between M and RME. There is a more elaborate version of this recommendation between Chapters 2 and 3 and a one-page summary of the issues covered at the beginning of each chapter.

# CHAPTER 1

## BACKGROUND AND CONTEXT

### SUMMARY

The main part of this book, Chapters 3 – 8, consists of dialogues between two characters M, a mathematician, and RME, a researcher in mathematics education. M and RME discuss issues regarding the learning and teaching of mathematics at university level. The starting points of their discussions are data samples that exemplify these issues. Both data samples and discussion between M and RME originate in data collected in the course of studies that I have been involved in since 1992. In this chapter I introduce the international scene of mathematics education research that the book aims to contribute to and then outline these studies. The presentation is in three parts.

***Part 1. TALUM: the Teaching And Learning of Undergraduate Mathematics.***

*TALUM*, is a relatively new and rapidly developing field of mathematics education research. As, particularly in the 1990s, mathematics departments started to respond to the decline in the number of students who opt for mathematical studies at university level the realisation that, beyond syllabus change, there is also the need to reflect upon tertiary pedagogical practice began to grow. The research on which this book draws was conceived and carried out with the aim to address this need in a systematic and original way.

***Part 2. A rationale for a certain type of TALUM research***

The research on which this book is based draws on several traditions of educational research and has the following characteristics: it is collaborative, context-specific and data-grounded and, through being non-prescriptive and non-deficit, it aims to address the often difficult relationship between the communities of mathematics and mathematics education. A fundamental underlying belief of this work is that development in the practice of university-level mathematics teaching is manageable, and sustainable, if driven and owned by the mathematicians who are expected to implement it.

***Part 3. The series of studies on which this book draws***

These aimed to explore students' learning in the first, and sometimes, second year of their undergraduate studies (mostly in Analysis, Linear Algebra and Group Theory and mostly through observing them in tutorials and analysing their written work); and, to engage their lecturers in reflection upon learning issues and pedagogical practice (mostly in individual or group interviews).

## 1. TALUM: A GENERAL INTRODUCTION

The area in which the book aims to contribute, the teaching and learning of mathematics at the undergraduate level, is a relatively new and rapidly developing field of mathematics education research. One rationale for this emergence of interest originates in the alarming decrease in the number of students who opt for exclusively mathematical studies beyond the compulsory level<sup>1</sup> (Hillel, 2001) – a decrease largely due to the alienation from the mathematics they experience at school (e.g. Smith, 2004; Boaler, 1998; Nardi & Steward, 2003). The number of mathematics graduates who choose to go into school teaching is also declining (French, 2004; LMS, 1995). This is partly due to the allure of careers in industry and IT (Hillel, 2001), but it is due also to the lacklustre and often demoralising learning experience that their university studies have been (Goulding et al, 2003; HEFCE, 1996; Burn & Wood, 1995; Kaput & Dubinsky, 1994; Harel & Trgalová, 1996)<sup>2</sup>.

The decline in the numbers of mathematics graduates and well-qualified mathematicians willing to become teachers is likely to affect the quality of young people's experience of learning mathematics at school in ways that limit their appreciation and enjoyment of the subject (Johnston, 1994; MA, 2005; Perkins, 2005<sup>3</sup>). In turn this is likely to reduce the number of students who are willing and able to study mathematics at university level (Dick & Rallis, 1991) – and so the spiral of decline is perpetuated.

Mathematics departments have begun to respond to this decline in numbers by attempting to improve the students' learning experience from a cognitive as well as a socio-affective point of view (Petocz & Reid, 2005). Holton (2001) gives numerous examples of this trend. As a first step, mostly in the 1990s, and in order to adjust to the learning needs of new student intakes, many university mathematics departments undertook modifications of the syllabus (e.g. Kahn & Hoyles, 1997, in the UK; Ganter, 2000, in the USA). Beyond syllabus change, there has also been recognition of the need to reflect upon pedagogical practice (Anderson et al, 2000; McCallum, 2003) – and UK researchers have demonstrated significant initiative in this area (e.g.: Mason, 2002; Burton, 2004; Rodd, 2002; Alcock & Simpson, 2001; Povey & Angier, 2004; Brown & Rodd, 2004; Jaworski, 2002; Nardi et al, 2005; Iannone & Nardi, 2005a).

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<sup>1</sup> See, for example, reporting of this in the London Mathematical Society Newsletter, e.g. (Falconer, 2005a & b; LMS, 2005a&c; Qadir, 2005).

<sup>2</sup> It is almost paradoxical that this decline has been occurring in parallel with an unprecedented interest in mathematics and mathematicians in the media – as a browsing of recent titles in fiction, biography, film, theatre and television confirms. The mathematical community is starting to see that in this widespread, yet often stereotype-ridden (Davis, 2006), attention mathematics has been conferred, lies an opportunity for showcasing the relevance and beauty of the subject (e.g. Farley, 2006) and, thus, improve recruitment of young, committed and able mathematicians.

<sup>3</sup> 'there aren't enough teachers to put across the beauty, joy and excitement of mathematics' as well as its wide and increasing use in public debate (p31) bemoans Marcus du Sautoy (2006), a mathematician at the forefront of the community's attempts to communicate with the media (LMS 2006b; Mann 2005a and b), and others, in this brief report. Reporting of these attempts in other outlets such as the *Notices* of the American Mathematical Society, *The Mathematical Gazette* etc has also increased.