2

Modelling Preliminaries

2.1 Why Modelling?

To solve fluid flow problems and fully determine the dynamics of the flow, including mapping the velocity field and the relation between prevailing pressure and flow fields, is possible only in the most simply constructed cases and mostly in the physical sciences [134, 193]. Fluid flow problems in biology, by contrast, are rarely simply constructed and can rarely be solved directly [34, 120]. The problem of flow in a tube, for example, has the simple "Poiseuille flow" solution when the tube is rigid, its cross-section is perfectly circular, the tube is long enough for flow to fully develop, and the fluid is a smooth "continuum" that has the simple rheological properties of a "Newtonian" fluid in which shear stress is related linearly to the velocity gradients [34, 120]. Barely any of these ideal conditions is met in biological problems involving flow in tubes, most notably the problem of blood flow in arteries, and particularly flow in coronary arteries, which is the subject of this book. Here the fluid is not a smooth continuum but a suspension in plasma of discrete red and other blood cells and, as we saw in the previous chapter, the system does not consist of a single tube but of many millions of tube segments that are joined together in a hierarchical tree structure. The segments are rarely long enough or perfectly circular to support fully developed Poiseuille flow, and the details of flow at their junctions are highly complicated and depend strongly on the exact geometry of each junction [122]. Furthermore, the precise branching structure of the vascular system of the heart cannot be mapped to the last detail to allow a mathematical solution of the flow problem. In fact, it is known that these details vary widely from one heart to another as much as do fingerprints from one individual to the next [228].

The purpose of the vascular system of the heart is to bring blood flow to within reach of every cell of the myocardium. Schematically, the vascular system has the hierarchical form of a tree structure (Fig. 1.6.1), with flow proceeding from the root segment of the tree to the periphery. Pressure at the base of the aorta, where the vascular trees of the left and right coronary arteries have their roots (Fig. 1.3.2), provides the driving force for this flow, but the relationship between this pressure and the ultimate flow at the delivering end of the two trees is everything but simple [98, 97]. Indeed, it is far from clear that pressure at the base of the aorta is the *only* driving force for coronary blood flow, nor is it clear that the resistance to flow, which this driving force must overcome, is limited to that of simple flow in a tube. Other mechanisms may be at play, and while some are known, their exact role in the dynamics of coronary blood flow is as yet not fully understood. Prominent among these are the rhythmic contractions of the myocardium with each pumping cycle and the consequent effect of these contractions on vessels that are totally imbedded within that tissue. It has been demonstrated that one effect of this so-called "tissue pressure effect" is to reduce or even reverse the flow in the main coronary arteries during the contracting (systolic) phase of the pumping cycle [101], but it is possible that this same effect may actually provide a pumping (driving) force for blood flow within the peripheral vessels near the delivering end of the tree. The cyclic compression of coronary vasculature by surrounding tissue also has a "capacitance" effect, namely a cyclic change in the volume of blood contained in the system. This effect plays an important yet unclear role in the dynamics of the coronary circulation, rendering the relation between driving pressure and delivering flow far less tractable [96, 97]. The same is true of the effects of wave reflections from a massive number of vascular junctions within the coronary network and the important yet unclear role which these play in the dynamics of the coronary circulation [219].

Direct measurements of pressure and flow within elements of the coronary network, to establish an empirical relation between them, are fraught with no less difficulty. While some measurements have been made successfully in isolated hearts [98, 97], access is possible only to larger coronary vessels at entry into the coronary network, becoming increasingly difficult with increasing "depth" into the network. Measurements *in vivo* are further hampered by the violent motion of the coronary vessels as the heart contracts and relaxes in its periodic pumping action. Thus, at best some access is possible to one end of the coronary circulation, but this can provide only a limited base for any conclusions because of lack of access to the distal end of the circulation. More precisely, flow measured at entry to the coronary tree does not usually represent flow at exit, because of capacitance and other effects mentioned earlier.

Modelling is thus a necessity rather than a luxury in the study of coronary blood flow. In the absence of adequate access to the system for direct observations or measurements of pressure and flow, the only prospect for a good understanding of the system is by using a model. The accuracy of the model can be improved by testing it against whatever data or observations are available, changing its design so as to produce closer agreement. The obvious and most important advantage of using a model is that its behaviour can be studied easily and more extensively than the actual system which it represents. Indeed a range of such models have been proposed in the past and we examine some of them subsequently, but the emphasis in this book is less on the models themselves than on the elements from which the models are constructed. The reason for this is that a model of the coronary circulation is only useful if it can be tested against some direct measurements. In fact, the model must be tailored to the type of measurements available, and as the nature and availablility of such measurements changes, so must the design and nature of the model to be used.

Our understanding of the dynamics of the coronary circulation is presently at its infancy. Indeed, in the clinical setting a purely *static* view of the system predominates, in which the concern is primarily with whether vessels are fully open or restricted by disease [127, 133, 73]. The reason for this viewpoint is not that the *dynamics* of the coronary circulation are thought unimportant in the clinical setting but that as yet we do not have a clear understanding or a clear model of these dynamics. The purpose of this book is to provide the student, researcher, or indeed clinician, with basic analytical and conceptual tools with which to explore and hopefully improve his or her understanding of the dynamics of the coronary circulation.

2.2 The "Lumped Model" Concept

The relation between pressure and flow in a tube depends on such properties of the tube as its diameter, length, and elasticity. It also depends on the form of the driving pressure, in particular whether the pressure is steady or pulsatile. The relation between pressure and flow in a vascular tree structure consisting of a large number of tube segments depends not only on all such factors in each tube segment but also on events at the junctions between tube segments and on how the properties of individual segments are distributed within the tree structure. The overwhelming complexity of this problem gives rise to the "lumped model" concept. Detailed analysis and results based on this concept are presented in subsequent chapters. Here we discuss only broadly the concept itself as a valid modelling strategy.

Essentially, in a lumped model the complex vascular structure of the coronary network is ignored and the network is replaced by a single tube having properties representative of the network as a whole. It is a variant of the more familiar "black box" concept, in which a complex system is enclosed by an imaginary box and only the relation between input and output from the box is examined to learn something about the characteristics of the system without delving into the complexity that produced these characteristics. In the coronary circulation the lumped model attempts to reproduce a relation between pressure and flow similar to that observed or measured in the physiological system but without going through the overwhelming task of determining how the relation unfolds through the complex structure of the coronary vascular network.

Of particular interest is the relation between pressure and flow at *input* to the system and pressure and flow at *output*. The reason for this is that while some direct measurements of pressure and flow are possible at input to the system, usually at the left or right main coronary arteries, no such measurements are possible at output, that is at the capillary end of the system. The output end of the coronary circulation is of course of particular clinical interest because it represents the ultimate function of the system, namely the delivery of blood to cardiac tissue. But at this end of the system flow is divided into many millions of capillaries in which neither the velocity nor the number of capillaries can be determined with sufficient accuracy to compute total output. A correct model of the system would thus provide a theoretical means of obtaining important information at output which is not available experimentally. However, the "correctness" of the model can ultimately be verified only by testing its results against some measurements from the physiological system. Thus, the modelling process becomes a highly intricate *iterative* process whereby the choice and values of model parameters are guided by a comparison of the results of the model with whatever direct measurements are available [110, 24, 115, 90, 98, 97].

Pressure and flow in the coronary circulation are highly pulsatile because of the pulsatile nature of the input driving pressure and because, in addition to this, much of the coronary vasculature is imbedded in cardiac muscle tissue and is subject to the effects of cyclic contraction of the cardiac muscle, so-called "tissue-pressure effects". Thus, pressure and flow at both ends of the system are time-dependent in the sense that they have cyclic waveforms. The waveforms are not the same at both ends, however. At any point in time within the oscillatory cycle, total inflow into the coronary system is not usually equal to total outflow because of the so-called "capacitance" effect. There is continuous change in the total volume of vascular lumen within the system during the oscillatory cycle. Therefore, some inflow may go towards "inflating" the system and will not contribute to outflow and, conversely, some outflow may be produced by "deflation" of the system rather than by direct inflow. While *average* flow must be the same at both ends of the system, that is, flow averaged over one or more cycles, a relation between average flow and average pressure does not feature the time-dependent characteristics of the system that actually contribute to that relation. Only events *within the oscillatory cycle* exhibit these characteristics, but the nature of these events is lost in the time-averaging process. For these reasons the main focus of lumped models has been on a *time-dependent relation between pressure and flow*, that is on the time course of that relation within the oscillatory cycle.

2.3 Flow in a Tube

At the core of almost every modelling scheme for coronary blood flow and for blood flow in general is the mechanics of flow in a tube. Indeed, the lumped model discussed in the previous section is based on the concept that flow through the complex vasculature of the coronary circulation can be replaced by flow in a single tube with "equivalent" properties. It is important, therefore, to outline the basic properties of flow in a single tube, which we do in this section. The validity of the basic premise of the lumped model concept, namely that flow in a complex system of vessels *can* be considered equivalent to flow in a single tube, can only be discussed in the context of each particular modelling scheme and is therefore deferred to subsequent chapters.

When fluid enters a tube, it does not simply slide along the tube as a bullet, because of a condition of "no-slip" that prevails at the tube wall [13, 34, 174, 71] whereby elements of fluid in contact with the tube wall become arrested there, forming a cylindrical layer of stationary fluid attached to the inner surface of the tube wall. As fluid progresses along the tube, the next layer of fluid adjacent to the first is slowed down by the stationary layer because of the viscosity of the fluid, and similarly, subsequent concentric layers of fluid that are further and further away from the wall are slowed down but to a lesser and lesser extent and are thus able to move more freely, fluid along the axis of the tube able to move the fastest (Fig. 2.3.1).

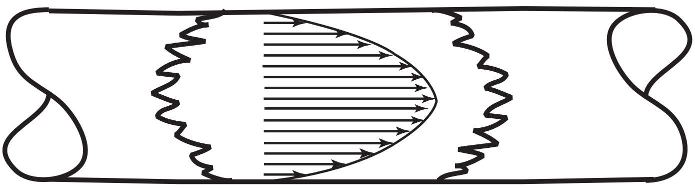

Fig. 2.3.1. Fully developed flow in a tube, commonly referred to as Poiseuille flow, is characterized by a parabolically shaped velocity profile, with zero velocity at the tube wall and maximum velocity along the tube axis.

Ultimately, at some distance downstream from the tube entrance, the flow becomes "fully developed" and is generally referred to as "Hagen-Poiseuille flow" after those who studied it first [168, 192, 174, 135], or more commonly as simply "Poiseuille flow". Flow in this region is characterized by a parabolically shaped "velocity profile" along a diameter of the tube, with zero velocity at the tube wall and maximum velocity at the tube axis, and is given by [221]

$$u = \frac{k}{4\mu}(r^2 - a^2) \tag{2.3.1}$$

where μ is viscosity of the fluid, r is radial coordinate measured from the axis of the tube, a is the tube radius, and k is the pressure gradient driving the flow, which in Poiseuille flow is constant and equal to the pressure difference Δp between any two points along the tube divided by the length of tube l between them, that is [221]

$$k = \frac{dp}{dx} = \frac{\Delta p}{l} \tag{2.3.2}$$

Here p is pressure and x is axial coordinate, positive in the direction of flow. The pressure difference Δp is measured in the direction of flow, that is

$$\Delta p = p_2 - p_1 \tag{2.3.3}$$

where p_1, p_2 are pressures at the upstream and downstream ends of the tube segment, respectively. Since p_1 must be higher than p_2 to produce flow in the positive $x-$direction, Δp is usually referred to as the "pressure drop" along the tube segment.

Eq. 2.3.1 indicates that in Poiseuille flow the flow rate q through the tube is given by

$$q = \int_0^a 2\pi r u dr = \frac{-k\pi a^4}{8\mu} \tag{2.3.4}$$

Thus, average flow velocity $\overline{u}$ is given by

$$\overline{u} = \frac{q}{\pi a^2} = \frac{-ka^2}{8\mu} \tag{2.3.5}$$

while maximum velocity $\hat{u}$ occurs on the tube axis where $r = 0$ and from Eq. 2.3.1 is given by

$$\hat{u} = \frac{-ka^2}{4\mu} \tag{2.3.6}$$

The two results show that maximum velocity in Poiseuille flow is twice the average velocity, that is

$$\hat{u} = 2\overline{u} \tag{2.3.7}$$

As described earlier, Poiseuille flow is not established immediately on entry into the tube, but evolves over a length of tube l_e known as the "entry length". Flow in that region of the tube is usually referred to as "developing flow" and an estimate of the entry length is given by [123, 174, 71]

$$l_e = 0.04 N_R d \tag{2.3.8}$$

where d is tube diameter and N_R is the Reynolds number, defined by

$$N_R = \frac{\rho \overline{u} d}{\mu} \tag{2.3.9}$$

where ρ is fluid density.

When the lumped model is used to study flow in the coronary circulation, which means that coronary blood flow is being modelled by an equivalent flow in a single tube, the equivalent flow is invariably considered *fully developed*. This assumption is fairly difficult to deal with because it is at once both necessary and unjustified. The assumption is unjustified because the entry lengths in many millions of tube segments in the coronary circulation will be different and cannot be represented by an "equivalent" entry length in a single tube. Furthermore, the assumption is necessary because the problem of determining the entry length and examining the extent to which flow is fully developed in each of these millions of tube segments is intractable. It is in fact further complicated because flow is entering and leaving tube segments at different stages of development. As a result, the standard entry length analysis leading to the result in Eq. 2.3.8, based on the assumption that flow entering the tube is uniform, no longer applies [31]. The best that can be done is to evaluate the weight of the assumption of fully developed flow in each modelling scheme in context of the particular aspect of coronary circulation being studied.

If flow entering a tube is assumed to have a uniform velocity $\overline{u}$, then a key difference between the developing and fully developed regions of the flow is that in the developing region elements of fluid near the tube axis (where $u = \hat{u}$) are being accelerated to meet the higher velocity there, while elements of fluid near the tube wall (where $u = 0$) are being decelerated because of the condition of no-slip at the tube wall. In the fully developed region, by contrast, fluid elements have reached their ultimate speed and are moving with constant velocity. This difference is compounded when the flow in a tube is *pulsatile*. In that case fluid elements in all regions of the tube are being accelerated and decelerated by the oscillatory driving pressure. Thus, in the entry region of the tube, fluid elements are being accelerated or decelerated *in space* by the entry conditions described above, and accelerated and decelerated *in time* by the oscillatory driving pressure. This makes the length of the entry region time-dependent and more difficult to define [34, 71, 7, 37].

2.4 Fluid Viscosity: Resistance to Flow

Flow in a tube may be resisted in a number of ways. If it is being accelerated, fluid inertia resists the pressure driving the flow. If the tube wall is elastic, its elasticity may oppose the driving pressure as it expands the tube wall. However, in both cases the same effect may also *aid* the flow, as it decelerates in the first instance, and as the tube wall recoils in the second. Thus, when flow in a tube is oscillatory these two forms of resistance do not dissipate energy, except in the second case if the tube wall is not purely elastic but has some viscoelastic properties.

The most important form of resistance to flow in a tube is that due to viscous friction at the interface between fluid and the tube wall. It is important because it is present when flow is steady or oscillatory and it always dissipates energy whether the flow is accelerating or decelerating. Because of this, it is usually referred to simply as "the resistance", and we shall follow this practice in this book. Resistance to flow in a tube arises because of a combination of the no-slip boundary condition at the tube wall and the viscous property of the fluid.

A key property of viscous fluids is that the force required to move adjacent layers of fluid at different velocities, that is, the force required to create shear flow, is an increasing function of the local velocity gradient. For a large class of fluids known as "Newtonian fluids", the force is simply proportional to the velocity gradient, that is

$$\tau = \mu \left(\frac{du}{dr} \right) \tag{2.4.1}$$

where τ is the local shear stress, that is, the local stress required to maintain the shearing motion, and μ is the coefficient of viscosity of the fluid. The velocity gradient du/dr is a measure of the local change in the velocity u of adjacent layers of fluid relative to the distance r between them. In Poiseuille flow this corresponds to the local slope of the parabolic velocity profile shown in Fig. 2.3.1 and given in Eq. 2.3.1.

The linear relation between shear stress and velocity gradient in Eq. 2.4.1 was first derived by Newton, hence the term "Newtonian fluids" has been used for fluids that obey the relation [168, 192]. There is a long-standing question whether blood, because of its corpuscular nature, is or is not a Newtonian fluid [21]. The question is not a very meaningful one because there are blood flow problems in which blood can be treated as a Newtonian fluid and others where it cannot. The question must therefore be directed at the nature of the flow problem being studied rather than at the nature of blood. Many problems relating to the general dynamics of flow in the systemic circulation, with focus on its pulsatile, have been studied successfully on the assumption of a Newtonian behaviour of the fluid, that is, on the assumption that Eq. 2.4.1 is valid [135, 141, 153]. That is not to say that blood *is* a Newtonian fluid, but that any non-Newtonian behaviour of blood does not significantly affect the general dynamics of the systemic circulation as a whole, although it may be important in the study of local flow properties in a single vessel or a single junction. The same is appropriate for a study of the general dynamics of the coronary circulation and we therefore uphold the Newtonian assumption in this book.

An important consequence of the viscous property of fluids is that the velocity difference between adjacent layers of the fluid must be infinitely small so that the velocity gradient remains finite. In other words, change of velocity within the fluid must be smooth. A step change of velocity (Fig. 2.4.1) is not possible because it would produce a locally infinite velocity gradient, and the shear stress required to maintain it would be infinite (Eq. 2.4.1).

It follows from this property that at the interface between a moving fluid and a solid boundary, as at the inner surface of a tube, there can be no finite difference between the fluid velocity tangential to the boundary and the

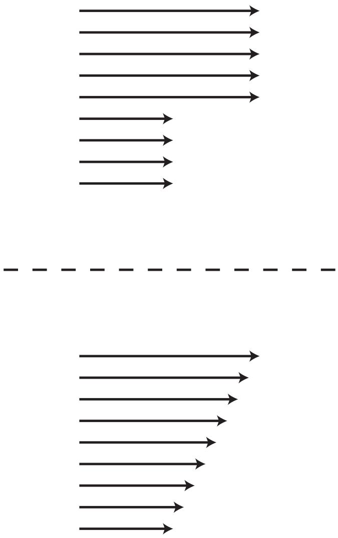

Fig. 2.4.1. An important consequence of the viscous property of fluids is that the velocity difference between adjacent layers of fluid must be infinitely small so that the velocity gradient remains finite. Thus, a step change of velocity (top) is not possible because it would produce a locally infinite velocity gradient and the shear stress required to maintain it would be infinite. Instead, the change of velocity must occur smoothly (bottom) so that the velocity gradient remains finite.

boundary itself. That is, the tangential velocity of fluid elements in contact with the boundary must be zero relative to the boundary, as required by the no-slip boundary condition (Fig. 2.4.2). This does not "prove" the no-slip boundary condition but shows only that the viscous property of fluids is consistent with it. Indeed, the basis of the no-slip boundary condition has been and remains largely empirical [13, 34, 174, 71].

Eq. 2.4.1 applied to Poiseuille flow in a tube, with velocity u as given by Eq. 2.3.1, yields the following result for the shear stress τ_w at the tube wall

$$\tau_w = \mu \left(\frac{du}{dr} \right)_{r=a} = \frac{ka}{2} = \frac{a\Delta p}{2l} \tag{2.4.2}$$

Since the pressure gradient k or pressure difference Δp are negative in the flow direction, it follows that τ_w is also negative. That is, the shear stress (acting on the fluid) at the tube wall has the effect of opposing the flow. The velocity gradient at the tube wall which is responsible for this shear stress is of course a consequence of the condition of "no-slip" there. It causes fluid in contact with the tube wall to come to rest while fluid along the tube axis charges at maximum velocity. A velocity gradient must therefore exist between the two regions and at the tube wall. Therefore, the condition of no-slip and the viscous property of the fluid *together* produce the shear stress at the tube wall.

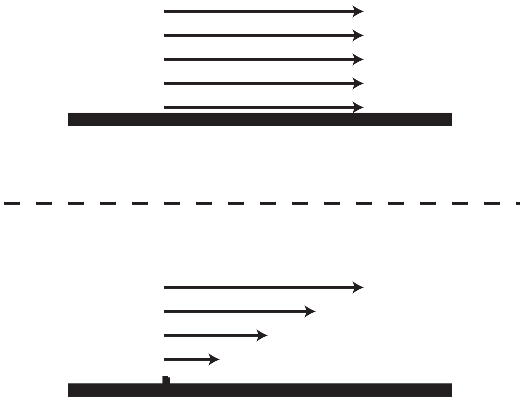

Fig. 2.4.2. The viscous property of fluids requires that at the interface between a moving fluid and a solid boundary, as at the inner surface of a tube, there be no finite difference between the fluid velocity tangential to the boundary and the boundary itself (top). That is, the tangential velocity of fluid elements in contact with the boundary must be zero relative to the boundary itself (bottom), as required by the no-slip boundary condition.

The total resistance to flow R, which results from shear stress acting on the entire surface area of the tube, can be expressed in terms of the flow rate q as

$$R = \frac{\Delta p}{q} \tag{2.4.3}$$

and substituting for the flow rate from Eq. 2.3.4, and using Eq. 2.3.2, this gives

$$R = -\frac{8\mu l}{\pi a^4} \tag{2.4.4}$$

The minus sign indicates that the resistance, which represents the force exerted by the tube wall on the fluid, is opposite to flow direction. The sign is usually omitted because the term "resistance" in fact refers to a force opposing the flow, that is a force in the negative direction when flow represents the positive direction. This is equivalent to modifying the definition of R to

$$R = -\frac{\Delta p}{q} = \frac{8\mu l}{\pi a^4} \tag{2.4.5}$$

It is seen that resistance to flow, which represents the amount of pressure difference required to produce a given amount of flow, depends critically on tube radius, being proportional to the inverse of the radius to the fourth power. Thus, if the tube radius is reduced by a factor of 2, the resistance increases by a factor of 16, that is by $1,600\%$. If the tube radius is *increased* by a factor of 2, the resistance decreases by a factor of 16, that is by approximately 94%.

Writing Eq. 2.4.5 as an equation for the flow rate q, we find the amount of flow that would be produced by a given pressure difference Δp, namely

$$q = -\frac{\pi a^4 \Delta p}{8\mu l} \tag{2.4.6}$$

If in an experiment the amount of flow is found higher than that dictated by Eq. 2.4.6, this could be interpreted as a change in one of the other parameters on the right side of the equation. Indeed, experiments in the past have shown that there is an *apparent* drop in blood viscosity in very small blood vessels, usually referred to as the Fahraeus-Lindqvist effect [63, 45, 221]. The effect is termed "apparent" because it is not based on direct measurement of the viscosity μ but on a measurement of flow for a given pressure drop. Thus, an observed value of q higher than that prescribed by Eq. 2.4.6 was interpreted as a decrease in the viscosity μ because such a decrease would also produce a higher value of q. Another interpretation which has been considered is the possibility of partial slip at the tube wall which would have the effect of requiring a smaller pressure drop for a given amount of flow, or conversely higher flow rate than is prescribed by Eq. 2.4.6, because of lower friction at the tube wall. However, it has been difficult to demonstrate that slip actually occurs in small blood vessels, and this interpretation is still a matter of debate [156, 211, 221]. Similar comments apply to the Fahraeus-Lindqvist effect because of the difficulties involved in actually measuring blood viscosity in small vessels. As a result of these difficulties it has not been possible, so far, to incorporate the concepts of slip or of the Fahraeus-Lindqvist effect into mainstream modelling schemes of the general dynamics of either the systemic or the coronary circulation.

2.5 Fluid Inertia: Inductance

Acceleration in fluid flow may occur in one of two ways: in space or in time. Acceleration in space occurs when the space available to a stream of fluid is decreasing, so the fluid must increase its velocity to go through a reduced amount of space. Flow in a tube with a narrowing, as in a bottle neck, is an example (Fig. 2.5.1). Velocity at the narrowing must be higher than it is elsewhere, since the flow rate through the tube must be everywhere the same by conservation of mass, and since it is assumed here that the flow is *incompressible*, that is fluid density is not changing. Thus, the fluid is in a state of acceleration as it goes through the narrowing. The acceleration is *in space*, that is, in the sense that fluid elements are being accelerated as they progress along the tube.

Another, less obvious, example of acceleration in space occurs at the entrance to a tube. If fluid enters with uniform velocity (Fig. 2.5.2), elements of the fluid along the tube axis must accelerate to meet the maximum velocity

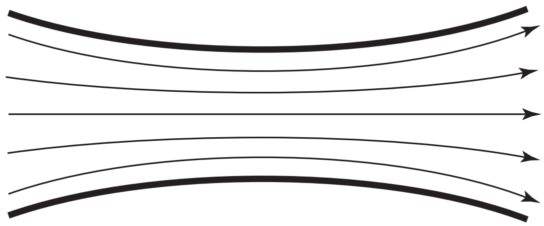

Fig. 2.5.1. Flow in a tube with a narrowing causes fluid elements to accelerate as they approach the narrowing and decelerate as they leave, assuming that the fluid is *incompressible*. Flow velocity is highest at the neck of the narrowing as indicated by the closeness of the streamlines there. Both the acceleration and deceleration are occurring *in space*, in the sense that the change in velocity is occurring as fluid elements progress along the tube.

in Poiseuille flow, while fluid elements near the tube wall are slowed down by the viscous resistance to meet the condition of no-slip at the tube wall. Thus in the entrance region of the tube some fluid is in a state of acceleration and some is in a state of deceleration, in both cases the change is occurring *in space*, that is as the fluid progresses along the tube.

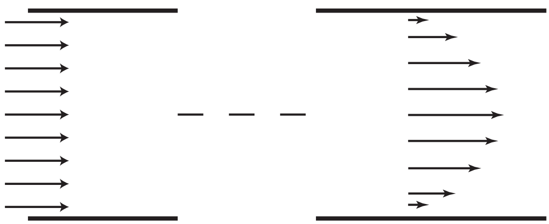

Fig. 2.5.2. Flow in the entrance region of a tube provides another example of acceleration and deceleration *in space*. If fluid enters with uniform veleocity, elements of the fluid along the tube axis must accelerate to meet the maximum velocity in Poiseuille flow, while fluid elements near the tube wall are slowed down by the viscous resistance and condition of no-slip at the tube wall.

One of the most important features of acceleration or deceleration in space is that it occurs in *steady* flow, that is, in a state of flow which does not change in time. In steady flow the velocity field does not change with time, meaning that the velocities at fixed positions within the flow field are constant and acceleration and deceleration occur as fluid elements move from one position to the next. It is in this sense that acceleration and deceleration in steady flow are seen as occurring *in space*.

Acceleration or deceleration *in time*, by contrast, is associated with *unsteady* flow, a state of flow in which the velocity distribution within the flow field changes with time. This situation occurs when the pressure driving the flow is not constant in time, as is the case in pulsatile blood flow where the driving pressure changes in an oscillatory manner (Fig. 2.5.3). In this case acceleration and deceleration are occurring *in time*, in the sense that the velocity at fixed points within the flow field is changing in time.

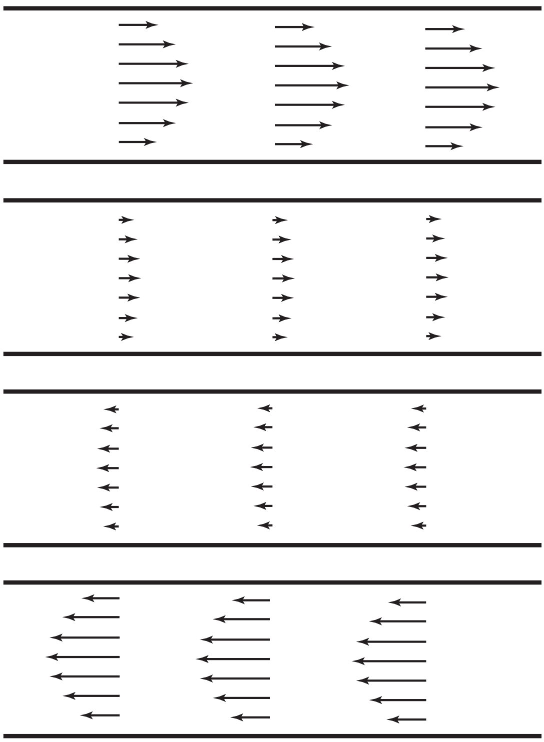

Fig. 2.5.3. Changing flow field in oscillatory flow. Different panels represent different points in time within the oscillatory cycle. Velocity is changing in time at fixed positions in space within the flow field. Acceleration and deceleration are occurring *in time*.

When a mass of fluid is accelerated or decelerated *in time*, the fluid does not respond immediately, because of its inertia. Thus, if the pressure difference Δp driving the flow in a tube changes suddenly to a higher level, it takes the flow rate q some time before it adjusts to a new value appropriate for the new driving pressure difference. This "reluctance" of the fluid to respond immediately is a form of resistance which would appropriately be referred to as "inertance" but is commonly known as *inductance* because of an electrical analogy to be discussed later.

Unlike the *viscous* resistance to flow which is present at constant flow rate, inductance is only present when flow is being accelerated or decelerated, that is, only when there is change in the flow rate. In fact, it is the *rate of* *change* of flow rate that is being resisted by the fluid, which means that a force is required to bring about such change. In the case of flow in a tube this means that a pressure difference Δp_L would be required specifically for this purpose; the subscript L is there to distinguish this pressure difference from that required to maintain the flow against the viscous resistance. More precisely, the required force is proportional to the rate of change of flow rate, that is

$$\Delta p_L = L\frac{dq}{dt} \tag{2.5.1}$$

Again, the symbol L is commonly used for the constant of proportionality because of analogy with inductance in electric systems.

The basis of this relation can be found in the mechanics of an isolated mass m, governed by Newton's law of motion, which asserts that the product of mass and acceleration must equal the net force acting on that mass. If the force is denoted by F and the position of the mass is denoted by x, the law can be written as

$$m\frac{d^2x}{dt^2} = F \tag{2.5.2}$$

where t is time. In general this equation is a vector equation because both F and x are vectors, but for the present purpose it is sufficient to work in only one dimension. In fluid flow the corresponding situation would be that of flow in a tube being accelerated, or decelerated, in one direction, namely along the axis of the tube. If the viscous effect at the tube wall is neglected for now (as it is accounted for separately below), then the body of fluid may be considered to move freely along the tube, as a bolus, in accordance with Newton's law. If the diameter of the tube is d, then the mass of such bolus of length l, being a cylindrical volume of fluid of diameter d and length l, is $\rho l\pi d^2/4$, where ρ is the density of the fluid. If the velocity of the bolus is u and the pressure difference driving it is Δp_L then the law of motion applied to this mass gives

$$\frac{\rho l\pi d^2}{4}\frac{du}{dt} = \Delta p_L\frac{\pi d^2}{4} \tag{2.5.3}$$

If q is the volumetric flow rate, then $q = u\pi d^2/4$ and the above can be put in the form

$$\Delta p_L = \left(\frac{4\rho l}{\pi d^2}\right)\frac{dq}{dt} \tag{2.5.4}$$

Comparison of this with Eq. 2.5.1 indicates that the constant L in that equation corresponds to the bracketed term above, that is

$$L = \left(\frac{4\rho l}{\pi d^2}\right) \tag{2.5.5}$$

Thus, Eq. 2.5.1 and the concept of inductance on which it is based have a basis in simple mechanics.

The total pressure difference Δp required to drive the flow in a tube in the presence of a change in flow rate is the sum of the pressure difference needed to overcome the force of resistance due to inductance, namely Δp_L, and the pressure difference needed to overcome the force of resistance due to viscosity discussed in the previous section, Eq. 2.4.3, now to be denoted by Δp_R, that is

$$\Delta p = \Delta p_R + \Delta p_L \tag{2.5.6}$$

Substituting for Δp_R from Eq. 2.4.3 and for Δp_L from Eq. 2.5.1, we then have

$$\Delta p = Rq + L\frac{dq}{dt} \tag{2.5.7}$$

This is a first order ordinary differential equation which has the general solution [116]

$$q(t) = \frac{e^{-t/(L/R)}}{L} \int \Delta p \; e^{t/(L/R)} dt \tag{2.5.8}$$

If the driving pressure difference is constant, say

$$\Delta p = \Delta p_0 \tag{2.5.9}$$

Eq. 2.5.8 gives upon integration

$$q(t) = \frac{\Delta p_0}{R} + Ae^{-t/(L/R)} \tag{2.5.10}$$

where A is a constant of integration. If the flow rate is zero at $t = 0$, we find $A = -\Delta p_0/R$ and the solution finally becomes

$$q(t) = \frac{\Delta p_0}{R}\left(1 - e^{-t/(L/R)}\right) \tag{2.5.11}$$

As time goes on, the exponential term vanishes, leaving the flow rate at a constant value of $\Delta p_0/R$, which is what it would be against a resistance R and with a driving pressure difference Δp_0 (Eq. 2.4.3). At that value the flow is said to be in *steady state*, while prior to that it is in a *transient state*.

The effect of inertia of the fluid is thus to cause the flow to take a certain amount of time to reach steady state. As the driving pressure difference is applied, the flow increases from zero to its ultimate value, but because of inertia it takes a certain amount of time to reach that value. The higher the inertial effect the longer it takes the flow to reach steady state (Fig. 2.5.4). The ratio L/R has the dimensions of time and is a measure of the time delay caused by the inertial effect. It is usually referred to as the "inertial time constant" and we shall denote it here by t_L, that is we define

$$t_L = \frac{L}{R} \tag{2.5.12}$$

The higher the value of t_L the higher the prevailing inertial effect and the longer is the time required for flow to reach steady state. It is important to note, however, that the approach to steady flow is *asymptotic*, as seen in Fig. 2.5.4, which means that, strictly, the flow takes an infinite amount of time to reach steady state. For practical purposes, however, the flow is sufficiently close to steady state in a finite and usually very short time. The inertial time constant t_L is a measure of that time. More precisely, if in Eq. 2.5.11 we write

$$\overline{q}(t) = \frac{q(t)}{\Delta p_0/R} \tag{2.5.13}$$

then

$$\overline{q}(t) = 1 - e^{-t/t_L} \tag{2.5.14}$$

and upon differentiation we find

$$\overline{q}\,'(t) = \frac{1}{t_L} e^{-t/t_L} \tag{2.5.15}$$

$$\overline{q}\,'(0) = \frac{1}{t_L} \tag{2.5.16}$$

Thus, the reciprocal of t_L represents the initial slope with which the flow curve moves towards its asymptotic value. The higher the inertial effect the higher the value of t_L and hence the lower the initial slope of the the flow curve and the longer it takes flow to reach its asymptotic value. Also, because the asymptotic value of the flow is here set at 1.0, then t_L also represents the time it takes the flow to reach this asymptotic value if, hypothetically, it continued with its initial slope, as illustrated in Fig. 2.5.4

It is important not to confuse transient and steady states here with developing and fully developed flow discussed in Section 2.3. Here, and essentially throughout the lumped model concept, the flow is assumed to be fully developed. Indeed, the relation $\Delta p_R = Rq$ used in Eq. 2.5.7 is based on the results obtained earlier for fully developed flow (Eq. 2.4.3). Steady and transient states here, by contrast, relate to flow development *in time*. Here we start out in a tube where fully developed flow is already established, then the pressure difference driving the flow is changed and we examine how, *in time*, the flow rate q adjusts to this change. Steady state is reached when the flow rate has fully adjusted to the change, while the adjustment period is referred to as the transient state. Thus, broadly speaking, developing and fully developed flow relate to flow development *in space*, as in the entrance region of a tube, while transient and steady states relate to flow development *in time*, as when the pressure difference driving the flow is changed.

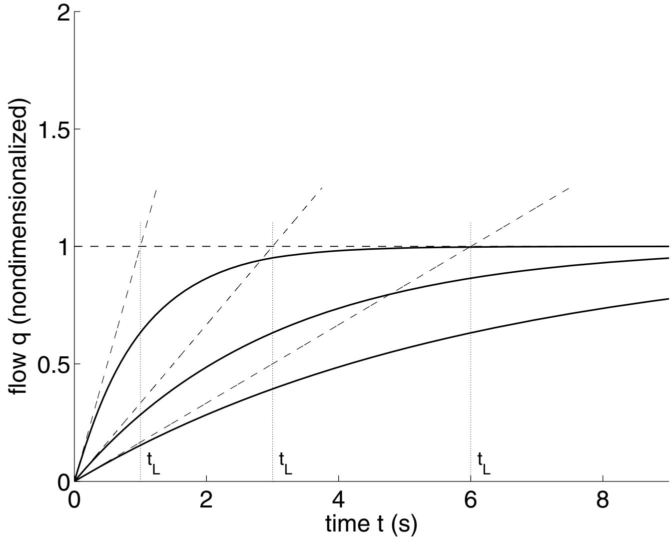

Fig. 2.5.4. If the pressure difference driving the flow in a tube is suddenly increased from 0 to some fixed value Δp_0, the flow increases gradually (solid curves) until it reaches the value $\Delta p_0/R$, which is shown by the dashed line above, normalized to 1.0. At that value the flow is said to be in *steady state*, while prior to that it is in a *transient state*. In steady state the flow rate has the value which it would have against a resistance R and with a driving pressure difference Δp_0 (Eq. 2.4.3), but because of fluid inertia the flow rate takes time to reach this value, the higher the inertia the longer the time. A good measure of the inertia of the fluid is the ratio L/R, which has the dimension of time when L is the inertial constant defined in Eq. 2.5.5 and R is the resistance defined in Eq. 2.4.4. The ratio is usually referred to as the "inertial time constant" and is denoted here by t_L (see Eq. 2.5.12). The three solid curves above, from left to right respectively, correspond to $L/R = t_L = 1.0, 3.0, 6.0$ *seconds*. It is seen clearly how the time it takes the flow curve to reach its ultimate value is directly related to the value of t_L. More specifically, the reciprocal of t_L represents the initial slope with which the flow curve moves towards its asymptotic value as indicated by the sloping dashed lines. The higher the inertial effect the higher the value of t_L and hence the lower the initial slope of the flow curve and the longer it takes the flow to reach its asymptotic value. Also, because the asymptotic value of the flow is here set at 1.0, then t_L also represents the time it takes the flow to reach this asymptotic value if, hypothetically, it continued with its initial slope. In the absence of the inertial effect ($L/R = t_L = 0$), the flow curve would "jump" to the asymptotic value at time $t = 0$ and remain on it thereafter.

If the driving pressure gradient $\varDelta p$ increases linearly with time, say

$$\varDelta p = \frac{\varDelta p_0}{T} t \tag{2.5.17}$$

where $\varDelta p_0$ is a constant and T is a fixed time interval, Eq. 2.5.8 gives upon integration (by parts) and simplification

$$q(t) = \frac{\varDelta p_0}{TR}\left(t - \frac{L}{R}\right) + Ae^{-t/(L/R)} \tag{2.5.18}$$

where A is a constant of integration. If the flow rate is zero at $t = 0$, we find $A = \varDelta p_0 L/(TR^2)$ and the solution becomes

$$q(t) = \frac{\varDelta p_0}{TR}\left(t - \frac{L}{R} + \frac{L}{R}e^{-t/(L/R)}\right) \tag{2.5.19}$$

or in nondimensional form

$$\overline{q}(t) = \frac{q(t)}{\varDelta p_0/R} = \frac{t}{T} - \frac{t_L}{T}\left(1 - e^{-(t/T)/(t_L/T)}\right) \tag{2.5.20}$$

It is clear from the form of the solution that the appropriate time variable in this case is the fractional time t/T, where T may, for example, be taken as the total interval over which the flow takes place, hence t/T has the range 0 to 1.0. As in the previous case, the effect of inertia is embodied in the value of t_L. Again, since t_L has the dimension of time, it is appropriate in this case to consider values of the inertial time constant t_L/T, as this indeed is the parameter required in the above equation.

Results for $t_L/T = 0.1, 0.3, 0.5$ are shown in Fig. 2.5.5. As the driving pressure difference $\varDelta p$ increases, the flow rate $\overline{q}\,'(t)$ begins to increase, but as in the previous case and because of inertia, it takes a certain amount of time for the flow to reach a value appropriate for the prevailing value of the pressure difference. But since in this case the pressure difference is continually increasing, the flow rate is never able to reach that appropriate value. What the flow rate is able to achieve as time goes on is a state in which its value is a *fixed amount* below what it should be. We may refer to this state as *quasi-steady state* since, strictly, steady state is usually defined as one in which the flow rate is either constant or periodic. In the present case it is continually increasing. Nevertheless, it is possible here to distinguish (Fig. 2.5.5) between an initial period where the flow rate is adjusting to the new pressure difference, which may be referred to as a transient state, and a final period in which the flow rate is still changing but is now changing at a fixed rate, the same rate at which the driving pressure difference is changing. It is in this sense that the latter may be referred to as quasi-steady state.

From Eq. 2.5.20 we see that the quasi-steady state is reached asymptotically, as the exponential term becomes insignificant, and the flow rate reduces to

$$\overline{q}(t) \sim \frac{t}{T} - \frac{t_L}{T} \tag{2.5.21}$$

Thus, asymptotically, the flow acquires the same form as the driving pressure, namely that of a linearly increasing function with a unit slope (Eq. 2.5.20), but, because of the inertial effect the flow curve is shifted along the time axis by an amount equal to the value of t_L/T as shown in Fig. 2.5.5. This shift represents the time interval by which the flow rate lags behind the prevailing pressure difference. The higher the inertial effect, the higher the value of t_L and the larger this ultimate gap between pressure and flow. Also, this gap between the flow and driving pressure never closes in this case because the driving pressure is continuouly changing. Only in the case of constant driving pressure does the flow ultimately "catch up" with the prevailing pressure and in a sense "overcome" the inertial effect as it reaches steady state. In the case of continuously changing pressure, as in the present case, the inertial effect is present in the transient as well as in the quasi-steady state.

If, finally, the driving pressure difference Δp varies as a *periodic* function of time, say

$$\Delta p = \Delta p_0 \sin \omega t \tag{2.5.22}$$

where ω is the angular frequency of the oscillation, then Eq. 2.5.8 gives upon integration (by parts again)

$$q(t) = \frac{\Delta p_0 (R \sin \omega t - \omega L \cos \omega t)}{R^2 + \omega^2 L^2} + A e^{-(R/L)t} \tag{2.5.23}$$

where A is a constant of integration. If the flow rate is zero at time $t = 0$, we find

$$A = \Delta p_0 \omega L / (R^2 + \omega^2 L^2) \tag{2.5.24}$$

and the solution becomes

$$q(t) = \frac{\Delta p_0}{R^2 + \omega^2 L^2} \left(R \sin \omega t - \omega L \cos \omega t + \omega L e^{-(R/L)t} \right) \tag{2.5.25}$$

A more useful form of the solution is obtained by combining the two trigonometric terms to give

$$q(t) = \frac{\Delta p_0}{\sqrt{R^2 + \omega^2 L^2}} \left(\sin(\omega t - \theta) - \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} e^{-(R/L)t} \right) \tag{2.5.26}$$

where

$$\theta = \tan^{-1} \left(\frac{\omega L}{R} \right) \tag{2.5.27}$$

or in nondimensional form

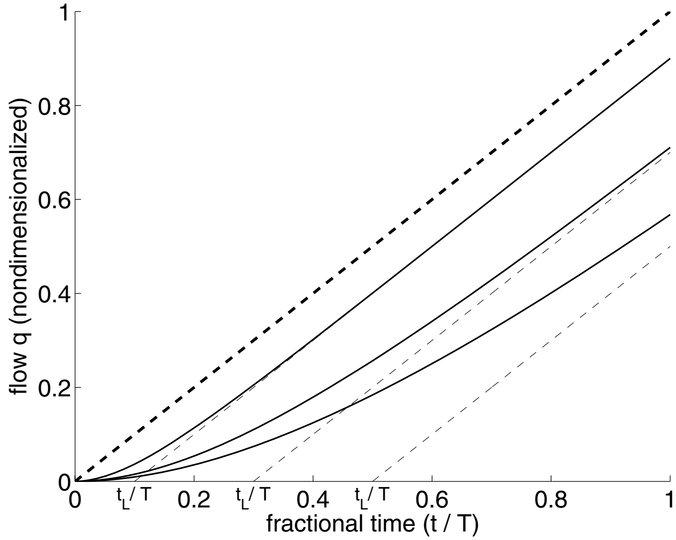

Fig. 2.5.5. If the pressure difference driving the flow in a tube increases linearly from zero, the flow rate begins to increase, but because of inertia it requires a certain amount of time to reach a value appropriate for the prevailing value of the pressure difference. But since in this case the pressure difference is continually increasing, the flow rate is never able to reach that appropriate value. What the flow rate is able to achieve as time goes on is a *quasi-steady state* in which its value is a fixed amount below what it should be. Thus, asymptotically, the flow acquires the same form as the driving pressure, namely that of a linearly increasing function with a unit slope (Eq. 2.5.20), but, because of the inertial effect the flow curve is shifted along the time axis by an amount equal to the value of t_L/T as shown. The three solid curves above, from left to right respectively, correspond to $t_L/T = L/RT = 0.1, 0.3, 0.5$, where T is total time interval over which flow is taking place, here taken as 1.0. The heavy dashed curve represents what the flow rate would be in the absence of inertial effect, that is when the inertial parameter t_L/T is zero. The light dashed curves represent the asymptotes of the flow curves for other values of the inertial parameter, shown at the bottom. It is seen that the higher the value of t_L/T the larger the ultimate gap between pressure and flow and hence the higher the inertial effect.

$$\overline{q}(t) = \frac{q(t)}{\Delta p_0/R} = \frac{1}{\sqrt{1+\omega^2 t_L^2}} \left(\sin(\omega t - \theta) - \frac{\omega t_L}{\sqrt{1+\omega^2 t_L^2}} e^{-t/t_L} \right) \quad (2.5.28)$$

$$\theta = \tan^{-1}(\omega t_L) \quad (2.5.29)$$

In this form we see that as the exponential term becomes insignificant, the flow rate becomes the same function of time as the oscillatory pressure difference, but with phase angle shift θ. The size of the shift is higher the higher the inertia of the fluid, that is the higher the value of the inertial time constant

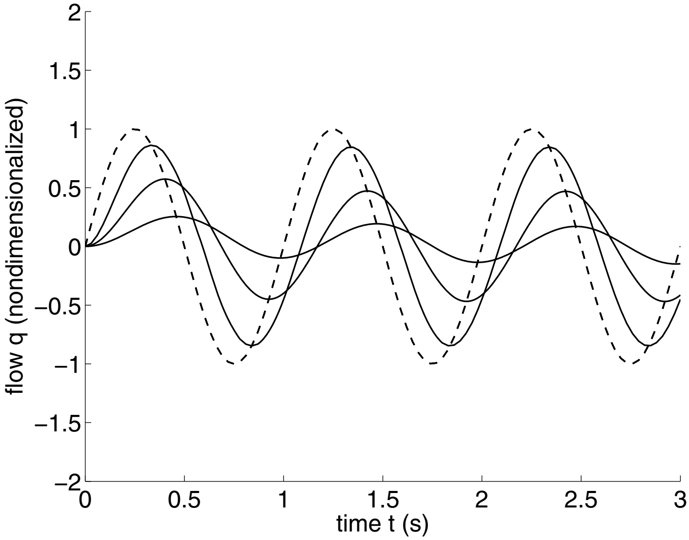

Fig. 2.5.6. If the pressure difference driving flow in a tube changes in an oscillatory manner, the flow rate attempts to follow the same oscillatory pattern, but because of inertia it requires a certain amount of time to reach that pattern. When it does, however, the flow rate lags behind the pressure difference by a fixed phase angle θ and its amplitude is lower than it would be in the absence of inertial effects, which here has the normalized value of 1.0. The three solid curves above, from left to right respectively, correspond to $t_L = L/R = 0.1, 0.3, 1.0$ *seconds*. It is seen that the higher the value of the inertial time constant t_L the larger the phase shift θ and the lower the amplitude of the flow oscillations.

$t_L (= L/R)$. Thus, here we see essentially the same behaviour of the fluid as in the previous case. The flow begins with a transient period in which it attempts to satisfy the prevailing pressure difference, but it never does. Instead, a steady state is reached in which the flow rate oscillates with the same frequency as the pressure difference driving the flow. It is a true "steady state" in this case, by common definition of that term [116]. In this state the flow rate oscillates in tandem with but lags behind the pressure difference by a fixed angle θ. The higher the inertial effect the larger is θ, and in the absence of inertial effects $\theta = 0$ as can be seen from Eq. 2.5.29. Also, from Eq. 2.5.28 we see that the *amplitude* of flow oscillation, which represents the highest flow rate reached at the peak of each cycle, is given by

$$|\overline{q}(t)| = \frac{1}{\sqrt{1 + \omega^2 t_L^2}} \tag{2.5.30}$$

thus the higher the inertial effect, hence the higher the value of t_L, the lower the amplitude of flow oscillation, as seen in Fig. 2.5.6. In the absence of inertial effects the amplitude of flow oscillation would be 1.0.

2.6 Elasticity of the Tube Wall: Capacitance

A tube in which the walls are rigid offers a fixed amount of space within it, hence the volume of fluid filling it must also be fixed, assuming, here and throughout the book, that the fluid is incompressible. By the law of conservation of mass, flow rate q_1 entering the tube must equal flow rate q_2 at exit. There is thus only one flow rate q through the tube, which may vary at different points in time depending on the applied pressure gradient, but at any point in time it must be the same at all points along the tube. Indeed, the relations between pressure gradient and flow considered in previous sections were all of this type, where the flow rate q may be a function of time t but not a function of position x along the tube (Eqs.2.3.4,2.4.6,2.5.11,14,19 and Figs.2.5.3-6). Thus, the analyses and results of previous sections were all based on the implicit assumption that flow is occurring in a rigid tube.

When flow is occurring in a *nonrigid* tube, two new effects come into play. First, the volume of the tube as a whole may change, an effect known as *capacitance*, again by analogy with the effect of a capacitor in an electric circuit. Second, a local change of pressure in an elastic tube *propagates* like a wave crest down the tube at a finite speed known as the wave speed. In a rigid tube, by contrast, a local change of pressure takes effect instantaneously everywhere within the tube. Consequently, the difference between flow of an incompressible fluid in a rigid tube compared with that in an elastic tube can also be expressed by saying that the wave speed is infinite in a rigid tube but is finite in an elastic tube.

While both the effects of capacitance and wave propagation result from elasticity of the tube wall, there is a fundamental difference between them, which provides a basis for dealing with them separately. Under the effect of capacitance there is a change in the *total volume* of the tube or system of tubes. Under the effect of wave propagation there is no change in the total volume of the system– a change of volume occurs only locally, as a local bulge or narrowing, and then propagates down the tube. It is important to emphasize, however, that while this difference makes it possible to separate the two effects on theoretical grounds, it does not necessarily imply that the two effects actually occur separately in practice. Hence, in this and the next section we deal with the effects of capacitance and wave propagation separately, with the understanding that this does not imply that the two effects must or do occur separately.

The key to the capacitance effect on flow in an elastic tube is that it affects the *total volume* of the tube, therefore flow rate at entrance to the tube may no longer be the same as that at exit because some of the flow at entry may go towards inflating the tube while some of the flow at exit may have come from a deflation of the tube. A convenient way of modelling this is to imagine flow going into a rigid tube to which a balloon is attached such that fluid has the option of flowing through the tube as well as inflating the balloon as depicted schematically in Fig. 2.6.1. The choice of a rigid tube is essential in order to eliminate the possibility of local changes in volume that would occur in wave propagation. Thus, the model depicts change in total volume only, consistent with the capacitance effect in isolation.

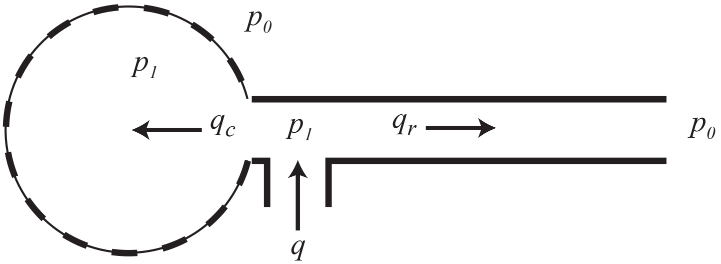

Fig. 2.6.1. Capacitance effect of flow in an elastic tube can be modelled by flow into a rigid tube with a balloon attached at one end. Flow rate q entering the system may go into the balloon or into the tube or both. Pressure p_1 at entry into the system is equal to pressure prevailing inside the balloon. Pressure at exit from the rigid tube is p_0, the same as that outside the balloon.

Initially, we consider the entrance to the balloon to be at entrance to the tube, so that pressure p_1 at entry into the system is equal to pressure prevailing within the balloon. Pressure outside the balloon and at exit from the tube is p_0. Flow through the tube and flow into the balloon are thus *in parallel*, in the sense that they can occur *independently* of each other.

Flow through the tube, to be denoted by q_r, is determined by the viscous resistance R and by the pressure difference Δp, as found previously (Eq. 2.4.3), namely

$$q_r = \frac{\Delta p}{R} \tag{2.6.1}$$

where

$$\Delta p = p_0 - p_1 \tag{2.6.2}$$

For flow into the balloon we note first that the balloon is in an inflated state when pressure inside the balloon is higher than pressure outside it, that is when

$$p_1 > p_0, \quad \Delta p < 0 \tag{2.6.3}$$

If the volume of the balloon in this state is v, then the capacitance C which is a measure of the *compliance* of the balloon is usually defined by the amount of *change* in the pressure difference Δp required to produce a change Δv in the volume of the balloon, that is

$$C = \frac{\Delta v}{\Delta(\Delta p)} \tag{2.6.4}$$

The notation in the denominator emphasizes that it is not the pressure difference Δp that produces the change in volume but a change in that pressure difference. Also, in this form it is seen that a higher value of C represents a balloon that requires less change in Δp to produce a given change in volume, that is a balloon that is more elastic, or more compliant.

In coronary blood flow and blood flow in general the change in volume Δv is not a useful entity to work with because it is not easily accessible. A more useful entity is the capacitive flow rate q_c representing the amount of flow going into or out of the balloon, which can be related to Δv in the following way. As before, we assume that fluid is incompressible, hence the only way to change the volume of the balloon is to change the amount of fluid within it, that is to have a nonzero flow rate q_c going into or out of the balloon. If a constant flow rate q_c occurs over a time interval Δt, the corresponding change in volume of the balloon will be

$$\Delta v = q_c \Delta t \tag{2.6.5}$$

Substituting this into Eq. 2.6.4 we then have

$$C = \frac{q_c \Delta t}{\Delta(\Delta p)} \tag{2.6.6}$$

therefore

$$q_c = C \frac{\Delta(\Delta p)}{\Delta t} \tag{2.6.7}$$

More generally, if Δp is a continuous function of time, then q_c correspondingly becomes a function of time, given by

$$q_c = C \frac{d(\Delta p)}{dt} \tag{2.6.8}$$

This result shows clearly, again, that flow rate into the balloon depends not on the pressure difference Δp but on the rate of change of that difference. Also, by noting that total flow rate q into the system must be the sum of flow rates into the balloon and the tube, that is

$$q = q_c + q_r \tag{2.6.9}$$

we see clearly that, because of the capacitance effect, flow rate q into the system is not necessarily equal to flow rate q_r out of the system.

If the pressure p_0 at exit from the tube and outside the balloon is now fixed, then flow into the system is controlled by only one remaining variable, namely the input pressure p_1. Under these conditions we consider the following three scenarios.

If the input pressure p_1 is constant, that is, if

$$\Delta p = \Delta p_0 \tag{2.6.10}$$

where Δp_0 is a constant, then Eqs.2.6.1,8 give

$$\begin{aligned} q_r &= \frac{\Delta p_0}{R} \\ q_c &= 0 \end{aligned} \tag{2.6.11}$$

Thus, in this case flow is entirely through the tube. Flow into the balloon is zero because the rate of change of Δp is zero (although Δp itself is not zero). The volume of the balloon remains unchanged in this case. The balloon comes into play only when Δp is a function of time, which occurs if p_1 is a function of time.

If, for example, p_1 increases linearly with time, then the pressure differences across the tube and across the balloon will also increase linearly with time, say

$$\Delta p = \frac{\Delta p_0}{T} t \tag{2.6.12}$$

where Δp_0 is a constant as before, t is time, and T is a fixed interval of time over which the change is taking place, which we introduce as in the previous section in order that Δp_0 retains the physical dimensions of pressure, then Eqs.2.6.1,8 now give

$$\begin{aligned} q_r &= \frac{\Delta p_0}{R} \frac{t}{T} \\ q_c &= C \frac{\Delta p_0}{T} \end{aligned} \tag{2.6.13}$$

There is constant flow into the balloon in this case, because the rate of change of Δp with time is constant. Flow through the tube increases linearly with time as Δp increases with time. To compare the two graphically it is easier to put them in nondimensional forms, namely

$$\begin{aligned} \overline{q}_r &= \frac{q_r}{\Delta p_0 / R} &= \frac{t}{T} \\ \overline{q}_c &= \frac{q_c}{\Delta p_0 / R} &= \frac{RC}{T} \end{aligned} \tag{2.6.14}$$

The product RC is seen to have the physical dimensions of time and is usually referred to as the "capacitive time constant". We shall denote it by t_c, in analogy with the inertial time constant (t_L), and define it by

$$t_c = RC \tag{2.6.15}$$

thus the two flow rates in nondimensional form are finally given by

$$\begin{aligned} \overline{q}_r &= \frac{q_r}{\Delta p_0/R} = \frac{t}{T} \\ \overline{q}_c &= \frac{q_c}{\Delta p_0/R} = \frac{t_c}{T} \end{aligned} \tag{2.6.16}$$

Fig. 2.6.2 compares these flow rates at different values of t_c. We recall that higher values of t_c $(= RC)$ are associated with higher compliance, allowing more flow to go into the balloon. Therefore, as seen in the figure, capacitive flow is constant at a value in fact equal to t_c/T, while resistive flow (flow through the tube) increases linearly as t/T.

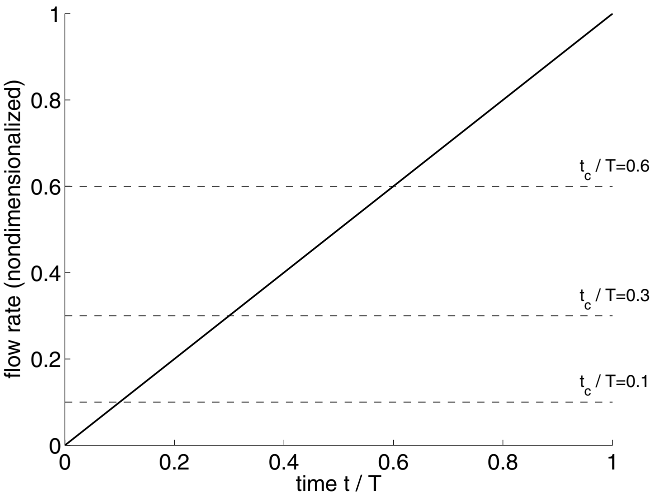

Fig. 2.6.2. Comparison of resistive (solid) and capacitive (dashed) flow rates when the driving pressure Δp is increasing linearly with time over a time interval T and at three different values of the capacitive time constant t_c. In all cases, capacitive flow is constant since it depends on the rate of change of Δp, while resistive flow increases linearly with time since it depends on Δp itself. Higher values of the capacitive constant t_c correspond to higher compliance, thus allowing more flow into the balloon.

Finally, an important scenario to consider is that in which the pressure differences across the tube and across the balloon is oscillatory, say

$$\Delta p = \Delta p_0 \sin \omega t \tag{2.6.17}$$

where Δp_0 is a constant and ω is the angular frequency of oscillation. In this case Eqs.2.6.1,8 give

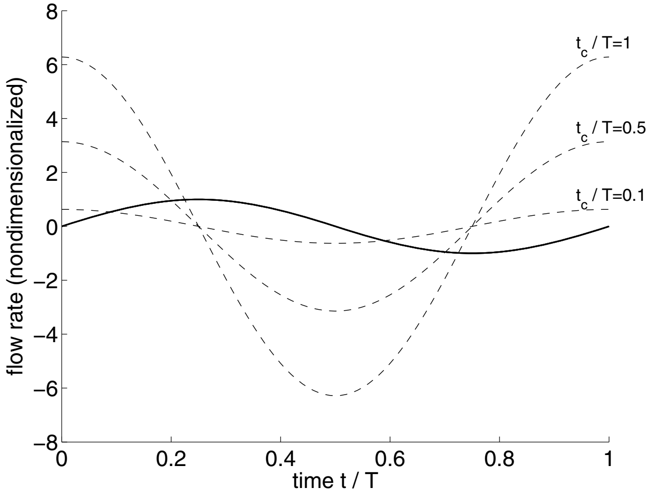

Fig. 2.6.3. Comparison of resistive (solid) and capacitive (dashed) flow rates when the driving pressure Δp is an oscillatory function of time of period T. The resistive flow (solid) has the same form as the driving pressure since inertial effects are not included here and since it is unaffected by the value of the capacitive time constant t_c. The capacitive flow (dashed) in each cycle, on the other hand, is higher with higher values of t_c because of higher compliance of the balloon.

$$
\begin{aligned}
q_r &= \frac{\Delta p_0}{R} \sin \omega t \\
q_c &= \Delta p_0 \omega C \cos \omega t
\end{aligned}
\tag{2.6.18}
$$

As expected, both q_c and q_r are oscillatory functions of time, with the same frequency as the driving pressure, namely ω. To compare the two it is more appropriate to put them in nondimensional forms, namely

$$
\begin{aligned}
\overline{q}_r &= \frac{q_r}{\Delta p_0 / R} &= \sin \omega t \\
\overline{q}_c &= \frac{q_c}{\Delta p_0 / R} &= \omega t_c \cos \omega t
\end{aligned}
\tag{2.6.19}
$$

The two flows are compared graphically in Fig. 2.6.3, where it is seen that how much of the flow goes into the balloon in each cycle depends on the value of the capacitive time constant t_C. As in the previous case, higher values of the t_c correspond to higher compliance, thus allowing more flow into the balloon. The resistive flow, on the other hand, is unaffected by the value of t_c and has the same form as the driving pressure, noting that inertial effects are not included here.

The results of this section illustrate the important role that capacitance plays in the dynamics of oscillatory flow in a compliant system, and hence its important role in the dynamics of the coronary circulation. While the structure of the coronary vascular system is far more complicated than the simple system in Fig. 2.6.1, the compliance of the system is known to play a role similar to that depicted in Fig. 2.6.1. A key question in the coronary circulation is how much of the oscillatory component of coronary blood flow goes into simply inflating and deflating the volume of the system, and how much goes into forward flow? This question is not properly addressed in the example of Fig. 2.6.1 because the driving pressure used here is a simple harmonic (Eq. 2.6.17) which produces only symmetrical back and forth flow in the rigid tube of Fig. 2.6.1. In coronary blood flow the driving pressure is a more complicated waveform which has a net forward component and some harmonic components. Because the forward and the oscillatory parts of the flow are not entirely separable from each other, capacitance of the system affects both, and much of the work in this subject is aimed at determining the nature and magnitude of this effect [98, 97].

2.7 Elasticity of the Tube Wall: Wave Propagation

As stated in the previous section, a fundamental difference between flow in a rigid tube and flow in an elastic tube is that a local change of pressure in a rigid tube is transmitted instantaneously to every part of the tube while in an elastic tube the change is transmitted with a finite speed. The reason for this is that a local increase in pressure in an elastic tube is able to stretch the tube wall outward, forming a local bulge, and when the change in pressure subsides, the bulge recoils and pushes the excess fluid down the tube [124]. The increase in pressure and the bulge associated with it propagate down the tube like the crest of an advancing wave. This scenario is not possible in a rigid tube because fluid in that case cannot stretch the tube wall, and because, as stated earlier, we assume throughout this discussion that the fluid is incompressible. It is for these two reasons that the local change in pressure in a rigid tube is transmitted instantaneously to every part of the tube. Wave propagation is not possible in a rigid tube.

If a change in pressure occurs at some interior position along an elastic tube, the change will propagate equally in both directions, towards both ends of the tube, as illustrated in Fig. 2.7.1. A scenario of more practical interest, however, is that in which a change in pressure occurs at one end of the tube and propagates in one direction towards the other end, which happens, for example, when a pump is placed at one end of a tube to drive the flow, or simply when there is a change in the pressure difference driving the flow. In this case wave propagation is in only one direction, namely from entrance to exit, as illustrated in Fig. 2.7.2, and this is the case we discuss in what follows under the general heading of wave propagation. However, the possibility exists that a wave propagating in one direction may be totally or partially reflected by an obstacle [221], thus leading to a secondary wave moving in the opposite

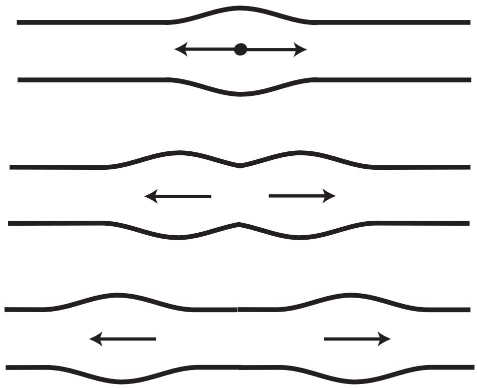

Fig. 2.7.1. A local change in pressure at an interior point in an elastic tube will propagate equally in both directions, towards the two ends of the tube.

direction as illustrated in Fig. 2.7.3. This will be discussed later in the book under the heading of wave reflections. Thus, in this section we consider only a primary wave moving from one end of an elastic tube to the other end.

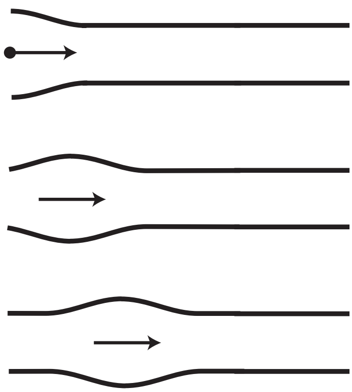

Fig. 2.7.2. A wave propagation scenario of more practical interest is that in which a change in pressure occurs at one end of a tube and propagates to the other end. This occurs, for example, when flow is driven by the stroke of a pump at the tube entrance, or simply when there is a change in the pressure difference driving the flow.

When considering flow in an elastic tube, it is useful to distinguish between *wave motion* and *fluid motion*. If the flow is driven by an increase in pressure at the tube entrance, for example, then wave motion refers to the forward motion of the local swelling or bulge in the tube caused by the increase in pressure, as illustrated in Figs. 2.7.2, 3, much like the motion of the crest of a wave on the surface of a lake. The speed at which the bulge advances along the tube is referred to as the *wave speed*. Fluid motion, on the other hand, refers to the motion of fluid elements within the tube, associated with that wave motion. As the wave crest passes each position along the tube, fluid elements at that location are first swept towards the local bulge in the tube, as illustrated schematically in Fig. 2.7.4, and then as the wave passes and the bulge subsides they are swept back by the decreasing pressure. The situation is again much the same as that experienced by a floating or submerged body swept by the passage of the crest of a wave on the surface of a lake.

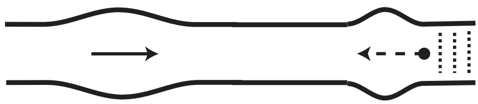

Fig. 2.7.3. A wave moving in one direction along an elastic tube may be reflected totally or partially by an obstacle, resulting in a secondary wave moving in the opposite direction.

The wave speed c in an elastic tube depends on the elasticity of the tube, a simple measure of that elasticity being the *Young's modulus* E, sometimes also referred to as the modulus of elasticity. The value of c also depends on the diameter d of the tube and its wall thickness h, and on the density ρ of the fluid. An approximate formula for the speed in terms of these properties is the so called Moen-Korteweg formula [168, 135, 34, 141]

$$c = \sqrt{\frac{Eh}{\rho d}} \tag{2.7.1}$$

The formula is only approximate because it does not take into account some dependence of the wave speed on viscosity of the fluid. Also, the formula is based on the assumption that the wall thickness h is small compared with the tube diameter. Despite these limitations the formula can be used to provide an estimate of the wave speed in the cardiovascular system. This is possible if it is further assumed that an *average* wall-thickness-to-diameter ratio h/d above can be taken for the entire system, which leaves c dependent on E and ρ only. Thus, taking $E = 10^7\ dyne/cm^2$, $\rho = 1\ g/cm^3$, and $h/d = 0.1$, we find $c = 1000\ cm/s$ which, in order of magnitude, is a representative estimate of the wave speed in the cardiovascular system.

If the pressure at the entrance of an elastic tube does not merely rise once but rises and falls in an oscillatory manner, the result is a *train* of wave crests moving in tandem along the tube, the distance between two consecutive crests being referred to as the *wave length* L, as illustrated in Fig. 2.7.5. Fluid motion within the tube then consists of back and forth movements everywhere along the tube as consecutive wave crests pass by. This situation provides a basic working model for flow in the cardiovascular system where the driving

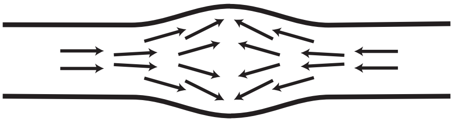

Fig. 2.7.4. As a wave crest passes each position along an elastic tube, fluid elements at that location are first swept towards the local bulge in the tube and then, as the crest passes and the bulge subsides, they are swept back by the decreasing pressure. This *fluid motion* is to be distinguished from the *wave motion*, illustrated in Figs. 2.7.1–3, which is concerned with only the motion of the wave itself. Fluid motion is shown above only schematically in order to illustrate the difference between fluid motion and wave motion, the motion of fluid elements is actually considerably more complicated.

pressure generated by the heart rises and falls in a periodic manner. If the *frequency* of oscillation is f *cycles/s* (Hz), then the wave length is related to the wave speed by

$$L = \frac{c}{f} \tag{2.7.2}$$

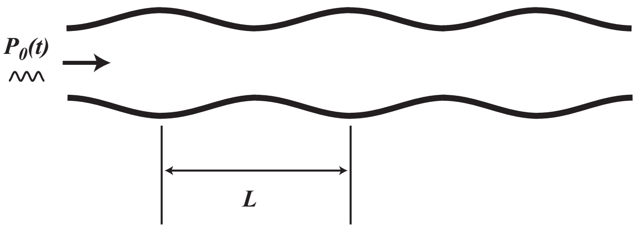

Fig. 2.7.5. If the pressure at the entrance of a tube does not change only once but continuously, in an oscillatory manner, the result is a train of wave crests moving along the tube, or what is commonly referred to as wave propagation. The distance L between two consecutive crests is referred to as the wave length.

If the frequency of oscillation of the pressure pulse generated by the heart is taken as 1 Hz, then an estimate of the wave length based on the above estimate of the wave speed is $L = 1000$ cm. The wave length is shorter at higher frequency, being only 500 cm at a frequency of 2 Hz. More important, the pressure pulse generated by the heart is actually a *composite wave* consisting of many so called *harmonic components*. Each harmonic component is a perfect *sine* or *cosine* wave but has a different amplitude and different frequency. The frequency of the pressure pulse generated by the heart represents only the so-called *fundamental frequency*, that is, the frequency of the first harmonic component which is referred to as the fundamental harmonic. The frequency of the second harmonic is double the fundamental frequency, and the frequency of the third harmonic is three times the fundamental frequency, and so on. Thus, the higher harmonics have increasingly shorter wave lengths.

Finally, wave speed and wave length are affected by the degree of elasticity of the vessel wall, via the value of Young's modulus E in Eq. 2.7.1. More rigid walls have higher values of E and therefore lead to higher wave speeds and higher wave lengths, which is relevant to blood vessels as they become more rigid, generally with age or more locally because of disease. In the limiting case of a totally rigid tube, E is infinite and hence both the wave speed and wave length become infinite. Wave propagation is therefore not possible in a rigid tube, clearly because a local increase in pressure cannot stretch the tube radially outward and thereby start the propagation process. Nevertheless, it is sometimes convenient to think of wave propagation in a rigid tube as one in which the wave speed is infinite, with a change in pressure at one end reaching all parts of the tube with infinite speed, that is instantaneously, as stated briefly in the previous section. Indeed, if the driving pressure at the entrance of a *rigid* tube changes in an oscillatory manner, the entire body of fluid within the tube oscillates back and forth in unison, which is not to be confused with wave propagation [221].

One of the most important effects of wave propagation in an elastic tube is the possibility of *wave reflections*. Wave reflections arise when a wave meets a change in one of the conditions under which it is propagating, such as the diameter or elasticity of the tube, or more generally any change in the resistance to wave propagation along the tube. It is important to distinguish between the *resistance to flow* in a tube and the *resistance to wave propagation* in that tube. The first represents the opposition to *flow* in a tube caused by the viscous shear at the tube wall, and is usually referred to as "pure resistance" or simply *resistance*. The second represents the opposition to *wave propagation* in a tube caused by a combination of elasticity of the tube wall and inertia of the fluid, and is usually referred to as *reactance*. We have noted earlier, for example, that wave propagation is not possible in a rigid tube. This can now be expressed more accurately by saying that a rigid tube has infinite reactance. More generally, a less elastic tube has higher reactance and offers more resistance to wave propagation than does a more elastic tube. The combined effects of reactance and pure resistance are commonly referred to as "impedance". We shall see later that wave reflections in a tube arise at a point where there is a change of impedance, which may be caused by a change of diameter or elasticity of the tube. Impedance and wave propagation play a central role in the dynamics of coronary blood flow and they are explored more fully in later chapters.

2.8 Mechanical Analogy

The mechanics of flow in a tube or a system of tubes can be identified, by analogy, with the basic mechanics of a solid object in motion under the influence of certain forces and conditions. Indeed, both situations are governed by the same laws of physics, and it should not be surprising that the analytical descriptions of their mechanics are analogous. What is different between the two situations, and what makes the analogy useful, stems from a difference not in the governing laws but in the type of forces and conditions involved and in the corresponding variables used in the two cases.

Thus, in the classical mechanics of a solid object, the familiar mass-damper-spring system is used in which an applied force may be opposed by a spring resistance proportional to the displacement of the object, a damper (or dashpot) resistance proportional to the rate of change of displacement (or velocity), and to an inertial resistance proportional to the second rate of displacement (or acceleration) [139, 76]. While in fluid flow these forces and conditions are not present in the same form, they are present in equivalent forms which obey the same governing laws, hence the basis for the analogy. For example, in fluid flow the capacitance of a tube or a system of tubes plays the role of the spring in the classical mechanics system, the viscous resistance between fluid and the tube wall plays the role of the damper, and the inertia of the fluid plays the role of the inertia of the solid object. These properties have already been discussed in earlier sections, what is required in this section is only to show how they translate into the properties of the classical mechanics system. The translation is not a direct one because the basic variables used in the classical mechanics system, namely mass, displacement and rates of displacement, are not readily available or convenient to work with in the fluid flow system.

Before we carry out this translation it is important to point out that the mechanical analogy has been used extensively in the modelling of coronary blood flow because the classical mechanics of a solid object are familiar and well understood. A model that can be expressed in terms of these mechanics, therefore, has the prospect of unveiling the unknown properties of the coronary circulation in terms which are familiar and well understood. In other words, the analogy is useful because the properties and behaviour of the mechanical system are more familiar and its elements more easily identified than the properties and elements of the fluid system. A potential for error is entailed in this modelling process, however, not because of any inaccuracy in the analogy but because elements of the coronary circulation required for the application of this analogy are not as easily identified as they are in a single tube. Thus, at the core of this modelling process is the fundamental "lumped model" assumption already discussed in Section 2.2, namely that the properties of many millions of tube segments in the coronary circulation can be represented collectively by those of an "equivalent" single tube. While many modelling studies have focused on the likely *values* of these lumped properties [111, 49, 59, 40, 121, 32, 102, 33, 195, 107, 98, 97]– capacitance, resistance, and inertance– the greater potential for error remains in the underlying assumption that these lumped properties actually exist. In other words, the mechanical analogy provides a mechanical model of the coronary circulation only on the assumption that the elements being modelled actually exist in the coronary circulation.

Furthermore, the behaviour of the classical mechanics system depends on a clear relation between the mass, the spring, and the damper. This relation is not known in the coronary circulation and must therefore be *assumed* in any modelling process. The effect of capacitance in the coronary circulation, for example, is produced by a change in the caliber, and hence the volume, of some coronary arteries, resulting in a change in the overall volume of the system [191, 51, 184, 110, 96, 97]. But at the same time this change in diameter also alters the resistance to flow in these vessels. The relation between these two effects is not known. In the classical mechanics system, by contrast, the elements representing capacitance and resistance are entirely separate and have no effect on each other. A related issue is the extent to which the basic elements of capacitance, resistance, and inertance are in series or in parallel in the coronary circulation. In the classical mechanics system this is known *a priori*, but not so in the coronary circulation. Some studies have attempted to deal with these issues by taking more than one lumped element of each type, that is, more than one resistance and more than one capacitance, for example, and by placing them in different combinations of series and parallel arrangements [24, 36, 91, 115].

Despite these difficulties, the mechanical analogy is a useful tool in modelling the coronary circulation because the analogy itself, as it applies to each individual element, is clearly valid. Thus, the relation between the flow rate q and pressure drop Δp in a tube, derived in Section 2.5 (Eq. 2.5.1), namely

$$\Delta p = L\frac{dq}{dt} \tag{2.8.1}$$

where L is the inertance, or inertial constant, of a bolus of fluid within the tube (Eq. 2.5.5), was shown in that section to be equivalent to the basic law of motion

$$F = m\frac{du}{dt} \tag{2.8.2}$$

where m is the mass of a solid object in motion,u is its velocity, and F is the force acting on it. The analogy between the two equations is apparent and the correspondence between the two situations is illustrated in Fig. 2.8.1. The driving pressure difference Δp in the fluid flow system corresponds to the acting force F in the classical mechanics system, while the inertance L corresponds to the mass m, and the flow rate q corresponds to the velocity u. In both cases the underlying law is "force equals mass times acceleration".

Similarly, the viscous resistance to flow in a tube, discussed in Section 2.4, and the resulting relation between the pressure difference Δp and the flow rate q, namely (Eq. 2.4.3)

$$\Delta p = Rq \tag{2.8.3}$$

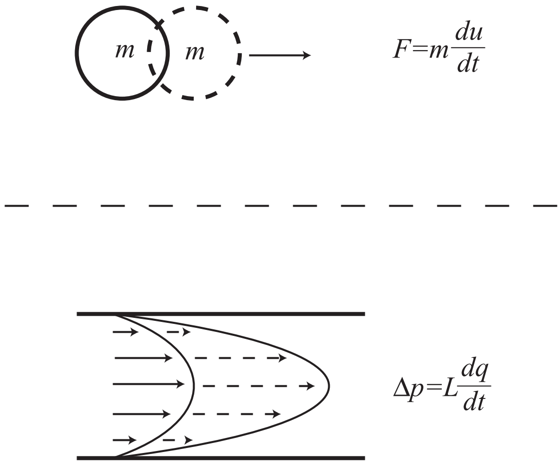

Fig. 2.8.1. The mechanical analogy between flow in a tube (bottom) and the motion of a solid object in classical mechanics (top). The driving pressure difference Δp in the fluid flow system corresponds to the acting force F in the classical mechanics system, while the inertance L corresponds to the mass m, and the flow rate q corresponds to the velocity u. In both cases the underlying law is "force equals mass times acceleration".

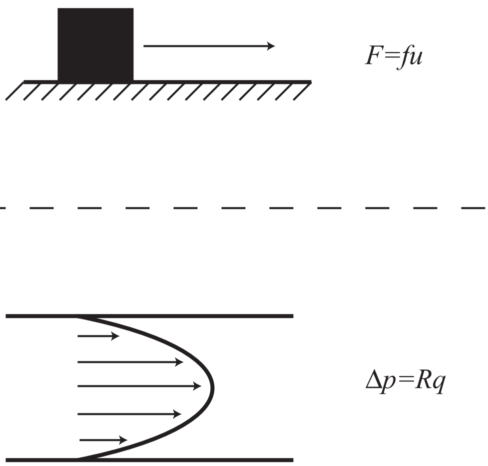

Fig. 2.8.2. Mechanical analogy between the viscous friction at the interface between fluid and tube wall, represented by velocity gradient at the tube wall (bottom), and the friction law in classical mechanics at the interface between two solid objects (top). Here the pressure difference Δp in the tube corresponds to the driving force F in the classical mechanics system, the flow rate q corresponds to the friction velocity u, and the viscous resistance R corresponds to the friction coefficient f.

where R is the resistance to flow due to viscosity (Eq. 2.4.4), is analogous to the classical law of friction at a solid-solid interface

$$F = fu \tag{2.8.4}$$

where f is the coefficient of friction at the interface, u is the relative velocity between the two surfaces, and F is the driving force. Again, the analogy between the two equations is apparent, and the two situations are illustrated in Fig. 2.8.2. Here the pressure difference Δp corresponds to the driving force F and the flow rate q corresponds to the velocity u, as before, and the viscous resistance R corresponds to the friction coefficient f.

Finally, the capacitance of an elastic tube, discussed in Section 2.6, and the resulting relation between the pressure difference Δp and the change in volume Δv, namely (Eqs. 2.6.4, 6)

$$\Delta(\Delta p) = \frac{1}{C}\Delta v \tag{2.8.5}$$

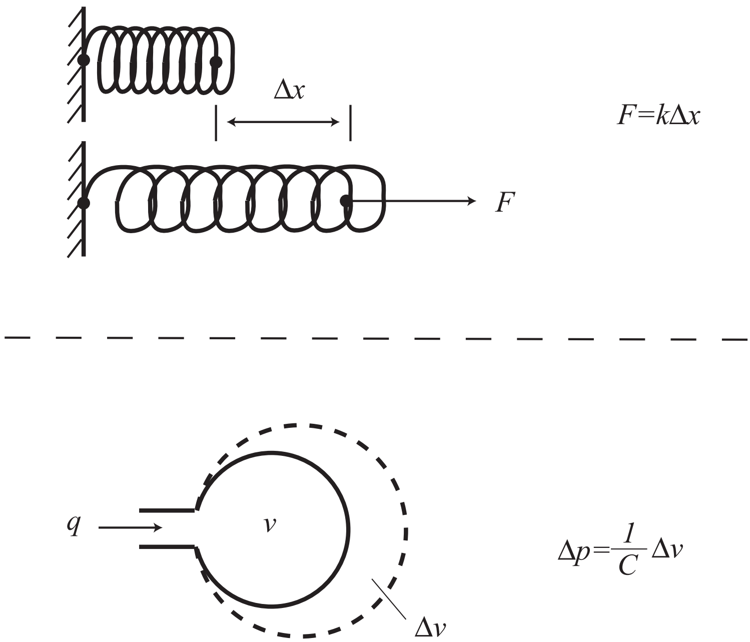

Fig. 2.8.3. Mechanical analogy: between the capacitance effect of flow in an elastic tube, here represented by a balloon, and the stretch of a spring according to Hooke's law. The pressure difference Δp in the flow system corresponds to the applied force F in the spring system, the change in volume Δv of the tube/balloon corresponds to the change in length Δx of the spring, and $1/C$ in the flow system corresponds to the spring constant k, where C is a measure of the compliance of the tube/balloon, as defined by Eq. 2.6.4.

$$= \frac{1}{C}\int q dt; \qquad \Delta v = \int q dt \tag{2.8.6}$$

where C is the capacitance of the tube, is analogous to the classical Hooke's law for an elastic spring, namely

$$F = k\Delta x \tag{2.8.7}$$

$$= k\int u dt; \qquad \Delta x = \int u dt \tag{2.8.8}$$

where k is the spring constant, Δx is the spring extension, and F is the applied force. In the integral terms above, the spring extension is expressed in terms of the velocity u with which the spring is being extended, and the change in volume Δv of the elastic tube/balloon is expressed in terms of the flow rate q. The analogy between the two equations is apparent, with Δp corresponding to the applied force F as before, the change in volume Δv in the flow system corresponding to the change in length Δx of the spring, and $1/C$ in the flow system corresponding to the spring constant k. In the integral terms the flow rate q is seen to correspond to the velocity u in the mechanical system, as in Eqs. 2.8.1, 2. The analogy between the two situations is illustrated in Fig. 2.8.3.

2.9 Electrical Analogy

The dynamics of the coronary circulation can also be modelled, by analogy, in terms of an electric circuit with the basic elements of resistance, capacitance, and inductance. This analogy is subject to the same limitations as the mechanical analogy discussed in the previous section, namely the assumption that these elements can be identified with lumped properties of the coronary circulation. Nevertheless, electrical analogies have been used extensively in the study of the coronary circulation [24, 36, 91, 115] because electric circuits are much easier to manipulate, both analytically and experimentally, and are thus a convenient modelling tool. A model of the coronary circulation based on the electrical analogy can actually be built and tested experimentally. This feature of the electrical model makes it particularly useful in the study of pulsatile flow.

In the electrical analogy the electric potential, or voltage, V corresponds to the pressure difference Δp in the flow system, and the electric current I along a conductor corresponds to the flow rate q along a tube. The basis of the analogy is that the relation between the voltage and current across an inductor L, namely [43]

$$V = L\frac{dI}{dt} \tag{2.9.1}$$

is analogous to the corresponding relation between the pressure difference and flow rate in a tube, as in Eq. 2.8.1, where the inertia of the fluid produces

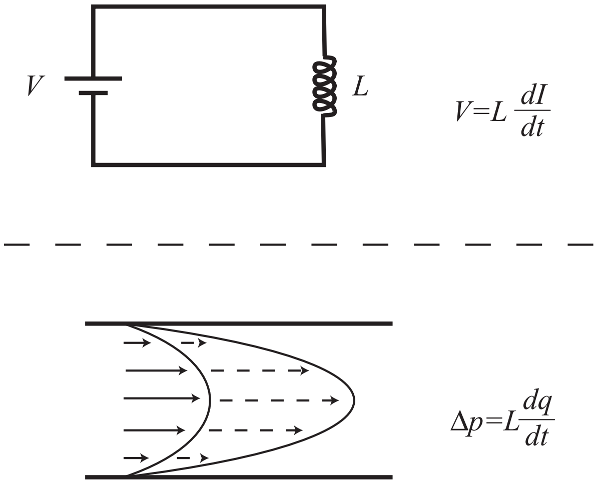

Fig. 2.9.1. Electrical analogy: between flow in a tube and the flow of current in an electric circuit, in the presence of inductance L in both systems. The driving pressure difference Δp in the fluid flow system corresponds to the voltage V in the electrical system, and the flow rate q corresponds to the electric current I. Inductance in the fluid flow system is caused by a change in the flow rate, which is associated with acceleration or deceleration of a mass of fluid, while inductance in the electrical system is due to change in the current, which is associated with acceleration or deceleration of a mass of electrons.

an effect analogous to that of an inductor, as discussed in Section 2.5. The analogy is illustrated in Fig. 2.9.1.

Similarly, the relation between the voltage and current across a resistor R, namely [43]

$$V = RI \tag{2.9.2}$$

is analogous to the relation between the pressure difference and flow rate in a tube, as in Eq. 2.8.3, where viscous friction between fluid and the tube wall produces an effect analogous to that of a resistor, as discussed in Section 2.4. The analogy is illustrated in Fig. 2.9.2.

Finally, the relation between the voltage across and current into a capacitor namely [43]

$$V = \frac{1}{C}\Delta Q \tag{2.9.3}$$

$$= \frac{1}{C}\int I dt; \qquad \Delta Q = \int I dt \tag{2.9.4}$$

where C is the capacitance and ΔQ the accumulated electric charge on the capacitor, is analogous to the relation between the pressure difference and flow rate into an elastic tube, as in Eqs.2.8.5,6. Here, because of the elasticity

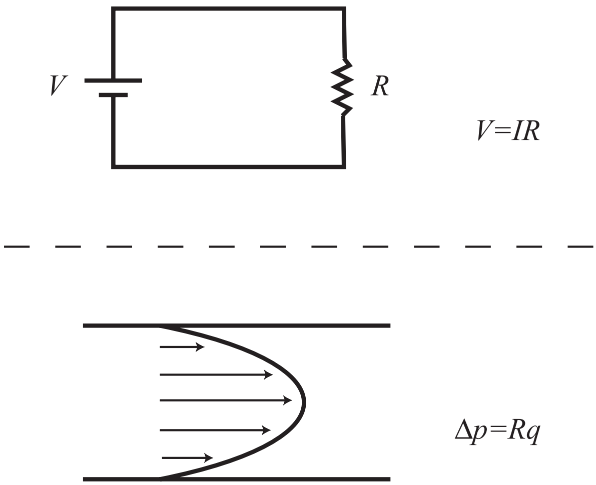

Fig. 2.9.2. Electrical analogy: between flow in a tube and the flow of current in an electric circuit, in the presence of resistance R in both systems. The driving pressure difference Δp in the fluid flow system corresponds to the voltage V in the electrical system, and the flow rate q corresponds to the electric current I. Resistance in the fluid flow system is due to loss of kinetic energy because of viscous friction between fluid and the tube wall, while that in the electrical system it results from a loss of electric energy within the resistor. Interestingly, in both cases the lost energy is converted to heat.

of the tube wall, the accumulated volume of fluid within the tube can change in analogy with a change in the electric charge accumulated on the capacitor. The analogy is illustrated in Fig. 2.9.3.

In summary, the electrical, mechanical, and fluid flow systems have three characteristics in common, namely inductance, resistance, and capacitance, and the dynamics of each system involves two principal variables, namely a driving force and consequent flow or motion. In the electrical system the three elements are an electric inductor, a resistor, and a capacitor, characterized respectively by their intrinsic constants L, R, C. The driving force is the voltage V, the motion is represented by the flow of electric current I, and the governing relations between these variables are:

$$\textit{inductance} \qquad V = L\frac{dI}{dt} \tag{2.9.5}$$

$$\textit{resistance} \qquad V = RI \tag{2.9.6}$$

$$\textit{capacitance} \qquad V = \frac{1}{C}\int I dt \tag{2.9.7}$$

In the mechanical system the three elements are a moving object, a damper, and a spring, characterized respectively by the mass m of the moving object, the friction constant f of the damper, and the spring constant k. The driving

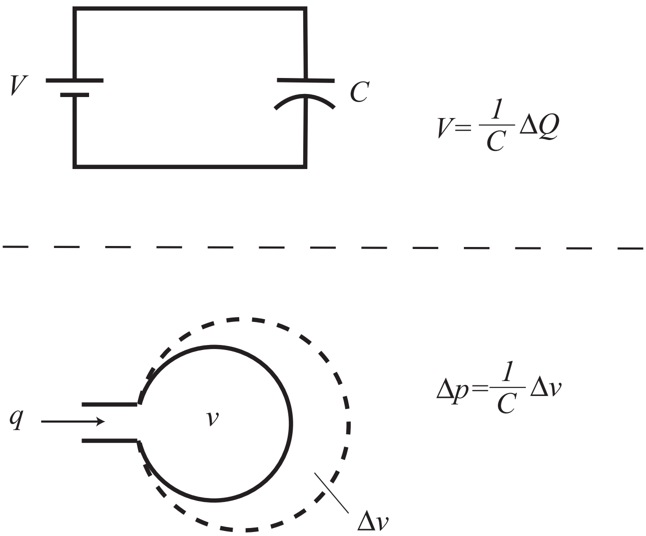

Fig. 2.9.3. Electrical analogy: between flow in a tube and the flow of current in an electric circuit, in the presence of capacitance C in both systems. The driving pressure difference Δp in the fluid flow system corresponds to the voltage V in the electrical system, and the flow rate q corresponds to the electric current I. Capacitance in the fluid flow system is due to a change in the volume v of fluid within an elastic tube, here represented by an expandable balloon, while in the electrical system it is caused by a change in the total electric charge Q on a capacitor. The change in volume Δv in the fluid flow system is attained by a sustained flow rate into or out of the balloon, while the change in electric charge ΔQ on the capacitor is attained by a sustained current into or out of the capacitor.

force is an applied force F, the motion is represented by the velocity u of the moving object, and the governing relations between these variables are:

$$\textit{inductance} \qquad F = m\frac{du}{dt} \tag{2.9.8}$$

$$\textit{resistance} \qquad F = fu \tag{2.9.9}$$

$$\textit{capacitance} \qquad F = k\int udt \tag{2.9.10}$$

In the fluid flow system, finally, the three elements are the mass of fluid in a tube, viscous resistance between moving fluid and the tube wall, and capacitance produced by the elasticity of the tube wall, characterized respectively by their intrinsic constants L, R, C. The driving force is the pressure difference Δp driving the flow, the motion is represented by the flow rate q, and the governing relations between these variables are:

$$\textit{inductance} \qquad \Delta p = L\frac{dq}{dt} \tag{2.9.11}$$

$$\textit{resistance} \qquad \Delta p = Rq \tag{2.9.12}$$

$$\textit{capacitance} \qquad \Delta p = \frac{1}{C}\int q dt \tag{2.9.13}$$

For flow in a tube of length l and radius a, assuming Poiseuille flow throughout, the resistance and inductance constants are respectively given by (Eqs. 2.4.4, and 2.5.5)

$$R = \left(\frac{8\mu l}{\pi a^4}\right); \qquad L = \left(\frac{\rho l}{\pi a^2}\right) \tag{2.9.14}$$

while the capacitance constant C is determined by the elasticity of the tube wall.

2.10 Summary

Modelling of the coronary circulation is necessary because experimental access to the *dynamics* of the system is severely limited. An understanding of the dynamics of coronary blood flow is important because in the absence of such understanding a purely *static* view of the system continues to be used in the clinical setting. In a static view of the system the primary concern is whether vessels are fully open or obstructed by disease. In a dynamic view the concern is more broadly based on all factors that may affects the dynamics of the system, the patency of the conducting vessels being only one such factor.

In a lumped model of the coronary circulation the complex vasculature of the system is essentially replaced by an "equivalent" single tube with "lumped parameters" that are assumed to represent the system as a whole. The model is tested against any measurements that can be obtained from the coronary circulation, and parameter values are adjusted in search of agreement. Despite difficulties associated with this concept, the lumped model has been an invaluable tool in the study of the coronary circulation by establishing some of its basic features.

The mechanics of flow in a tube is at the core of all lumped (as well as unlumped) model analysis. The analysis is usually based on the assumption of *fully developed* flow, ignoring flow in the entrance region of the tube where flow is in a developing phase. This assumption is fairly difficult to deal with because it is necessary, yet not easily justified.

Fluid viscosity together with the condition of no-slip at the tube wall produce "resistance" to *steady* flow in a tube. This resistance increases as the inverse of the tube radius to the fourth power, which means that if the radius of the tube is reduced by a factor of 2, the resistance to flow increases by a factor of 16. This dramatic relationship between vessel radius and resistance to flow figures heavily in clinical practice. It must be remembered, however, that coronary blood flow is not steady but pulsatile, where other forms of resistance exist.

When fluid is accelerated or decelerated, fluid *inertia* gives rise to another form of resistance to flow, commonly referred to as inductance. The immediate effect of inductance is to delay the response of the fluid to a change in the driving pressure difference. The flow rate does not "match" the prevailing pressure difference immediately but with a time delay. In that "transient state" the flow rate is attempting to reach a value appropriate for the prevailing pressure difference, and it ultimately does if the prevailing pressure difference does not change any further. But if the driving pressure difference continues to change, as in oscillatory flow, the flow rate never reaches that appropriate value. It falls short and lags behind, more so at higher values of the inertial constant.

The "capacitance" of a tube or system of tubes arises when the tube wall is elastic (or possibly viscoelastic) and hence the volume of fluid contained within the tube or system of tubes can change. Capacitance is known to play a significant role in the dynamics of coronary blood flow but a definitive model of that role has yet to be formulated.

In addition to giving rise to capacitance, another fundamental consequence of elasticity of the tube wall is that of wave propagation. In an elastic tube, a change of pressure at one end of the tube does not reach the other end instantaneously as it does in a rigid tube. Instead, it stretches the elastic wall of the tube locally at first and then propagates down the tube like the crest of a wave on the surface of a lake. In pulsatile flow this wave propagation is continuous in space (along the tube) and in time. Inductance of the fluid and capacitance of the tube combine to form a new type of resistance, namely "resistance to wave propagation", usually referred to as "reactance", to be distinguished from "pure resistance" caused by viscous shear at the tube wall. Reactance and pure resistance combine to form the "impedance", which plays a key role in "wave reflections", all of which will be discussed more fully in later chapters.

Flow in an elastic tube is governed by the same physical laws and the same equations as the motion of a mass in a mass-damper-spring system. By this so-called "mechanical analogy", inertia of the fluid in the fluid flow system is equivalent to the inertia of the mass in the mechanical system, viscous resistance in the fluid flow system is equivalent to resistance due to damper friction in the mechanical system, and capacitance of the tube in the fluid flow system is equivalent to the stretch of the spring in the mechanical system. The analogy is useful because the elements of the mechanical system are more familiar and their functions can be visualized more clearly than those in the fluid flow system.

Flow in an elastic tube is also analogous to the flow of current in an electric circuit. This so-called "electrical analogy" has been used widely in studying the dynamics of the coronary circulation and forms the basis and "language" of many lumped models of the system. Indeed, the terminology used for elements of resistance, inductance, and capacitance in the fluid flow system has been taken directly from these familiar elements in electrical systems, thus correspondence between the two is fairly clear. A great advantage of the electrical analogy is the availability of well developed mathematical analysis of electric circuits of a wide range of complexity which would be fairly difficult to formulate in terms of either the fluid flow or the mechanical system.